Numerals and their modifiers:
How morphology constrains alternatives

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Abstract Bare numerals (*three*), comparative-modified numerals (*more/less than three*), and superlative-modified numerals (*at most/least three*) differ in crucial ways with respect to bounding entailments, scalar implicatures, ignorance implicatures, and acceptability in downward-entailing environments. Many important strides have been made in deriving various subsets of their patterns, yet we still lack a unified account. In this paper I show that the key to such an account lies with the proper understanding of the contribution of the morphological pieces of these items – the numeral, *much/little*, and the comparative/superlative morpheme. From this we can not only obtain truth conditions that straightforwardly capture just the right bounding entailments for each item, but we can also naturally derive scalar and domain alternatives that yield just the desired scalar implicature, ignorance implicature, and acceptability in downward-entailing environments patterns. The account naturally shares features with existing alternative-based accounts of numerals, especially that of Spector (2015), but improves both empirically and conceptually on them all.

Keywords numerals, comparative, superlative, extents, scalar alternatives, domain alternatives, exhaustification, polarity

1 Introduction

Bare numerals (*three*; henceforth, BNs), comparative-modified numerals (*more/less than three*; henceforth, CMs), and superlative-modified numerals (*at most/least three*; henceforth SMs) are similar and different in interesting ways. There have been many theories trying to capture their behavior. At the same time I believe that none of the existing theories gets right all of the basic features of these items that we may like to understand. In this paper I am therefore going to make a new attempt. While the resulting theory naturally builds on many of the existing proposals and, most closely on Spector (2015), as I will show, it goes beyond them all both in terms of empirical coverage and theoretical economy.

The paper is structured as follows: In the rest of the introduction I examine the patterns of BNs, CMs, and SMs with respect to major phenomena such as entailments, scalar implicatures, ignorance, and acceptability in downward-entailing environments, concluding with a summary of the minimal desiderata for a theory of numerals that arise from them. In §2 I propose a unified theory of bare and modified numerals that meets all these desiderata (and more), deriving all the behavior of these items from their morphological pieces, on the one hand, and general and independently motivated pragmatic mechanisms, on the other. The theory uses both scalar and subdomain alternatives, and brings out a formal similarity between bare and modified numerals, on the one hand, and the polarity system, on the

other. In §3 I show that this theory outperforms existing theories of numerals both in terms of empirical coverage and in terms of theoretical economy. In §4 I present the conclusions as well as some of the open issues.

1.1 Entailments

Utterances of the form *three P Q*, *more than three P Q*, and *at least three P Q*, (1-a), all entail a lower bound. This can be seen from the fact that they all trigger negative inferences about values below the asserted value, (1-b), and these negative inferences are non-cancellable, (1-c).

(1)   a.   Alice has three / more than three / at least three diamonds.
         b.   ¬ The number of diamonds that Alice has is two or less / three or less / two or less.
         c.   Alice has three / more than three / at least three diamonds, # if not less.

In contrast, utterances of the form *less than three P Q* and *at most three P Q*, (2-a), both entail an upper bound (van Benthem 1986, Krifka 1999, Geurts and Nouwen 2007, Buccola and Spector 2016, a.o.).[1] We can see this from the fact that they both trigger negative inferences about values above the asserted value, (2-b), and that, just as before, these negative inferences are non-cancellable, (2-c).

(2)   a.   Alice has less than three / at most three diamonds.
         b.   ¬ The number of diamonds that Alice has is three or more / four or more.
         c.   Alice has less than three / at most three diamonds, # if not more.

1.2 Scalar implicatures

In an out-of-the-blue unembedded context, an utterance of *three P Q* is most naturally interpreted as *exactly three P Q*. This points to the fact that in addition to the lower-bounding inference *not less than three P Q* that we mentioned above, such an utterance also carries an upper-bounding inference *not more than three P Q*, (3-b). Unlike the lower-bounding inference, this upper-bounding inference is cancellable, (3-c).

(3)   a.   Alice has three diamonds.
         b.   ¬ Alice has four or more diamonds.
         c.   Alice has three diamonds, if not more.

     Since Horn (1972), the standard way to interpret the cancellability, i.e., optionality of the upper-bounding inference of *three P Q* is to treat it as a scalar implicature.[2] The analysis of bare numerals

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[1]Buccola and Spector (2016) argue that this upper-bounding entailment is there only for distributive predicates, but not for non-distributive predicates. I discuss this further in §3.13.

[2]The idea that bare numerals entail a lower-bounded meaning and acquire their upper bound via scalar implicature has been challenged recently. As in this paper I will uphold the traditional scalar implicature story, it is important to get a sense of the main objections against it and also of a potential response to those objections from the point of view of a scalar implicature account.

    First, it has been argued that another way to capture the optionality of the upper-bounding inference of bare numerals would be to say that bare numerals are *lexically* ambiguous between an 'at least' and an 'exactly' meaning. This is the approach taken, for example, by Geurts (2006), Kennedy (2013, 2015), or Nouwen (2010). Note however that such an approach is unable to derive the indirect scalar implicatures of bare numerals, for example, the fact that *If you read three books you are smart* implicates *but not if your read two*. That is because these implicatures are not about producing the exact meaning of n but rather about $n-1$. Note that, in contrast, indirect implicatures are completely expected on the standard view. (See (8) for a

that emerges from this is as follows: *Three P Q* entails *at least three P Q*. *Three* is a scalar item that belongs to a conventional scale of numerals. An utterance of *three P Q* thus activates alternatives based on other numerals, e.g., *two P Q* (= *at least two P Q*), or *four P Q* (= *at least four P Q*), etc. Upon hearing *three P Q*, one reasons that, say, *four P Q* would have been a stronger relevant statement to make, so through pragmatic reasoning reaches the conclusion that the reason the speaker didn't make that stronger relevant statement is because the speaker is not convinced that that statement is true ($\neg Bel_S \phi$ 'it is not the case that the speaker believes ϕ') or even believes it to be false ($Bel_S \neg \phi$ 'the speaker believes that it is not the case that ϕ').

Now, as Krifka (1999) points out, if numerals activate scalar alternatives, they should do so wherever they show up, that is, in modified numerals also. We would thus expect not only *three P Q* to activate scalar alternatives, but also *more/less than three P Q* and *at most/least three P Q*, as in (4).

(4)   **Predicted scalar alternatives of BNs, CMs, and SMs**
   a.   ScalAlts(*three P Q*) = {..., *two P Q, four P Q,* ...}
   b.   ScalAlts(*more/less than three P Q*) = {..., *more/less than two P Q, more/less than four P Q,* ...}
   c.   ScalAlts(*at most/least three P Q*) = {..., *at most/least two P Q, at most/least four P Q,* ...}

As a consequence of such alternatives, not only *Alice has three diamonds* should trigger scalar implicatures, but also the corresponding CM- and SM-sentences. However, they don't, or else, for example, we should be able to say *Alice has more than three diamonds* to mean that she has exactly four (more than three and not more than four = exactly four), which is clearly not how we use such utterances. (Below, '⤳' stands for 'implicates'.)

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more explicit example.) Conclusion 1: In order to capture not only the 'at least' and 'exactly' meaning of bare numerals but also their indirect implicatures one needs an implicature account.

Second, it has been argued that implicatures are traditionally a root phenomenon, but 'the exactly' meaning of bare numerals can appear in embedded contexts also. Kennedy (2013), for example, points out that *Neither of them read three of the articles on the syllabus; Kim read two and Lee read four* is coherent only if *three* means 'exactly three'. This objection can be easily overcome by using an updated implicature calculation system such as the one provided by grammatical theory of implicatures, which can derive embedded implicatures. Conclusion 2: In order to capture the 'exactly' meaning of bare numerals in embedded contexts one needs the grammatical theory of scalar implicatures.

Third, it has been argued that there are important differences between the strengthened meaning of bare numerals and the strengthened meaning of other scalar items. Kennedy (2013), for example, shows that, while (as we saw) the strengthened meaning of bare numerals is available in downward-entailing environments, for other scalar items it is almost impossible. On the experimental side, Marty et al. (2013) show that under memory load speakers err toward the strengthened ('exactly') reading for numerals but not for another scalar item such as *some*, and Huang et al. (2013) show that when given an unconscious choice both adults and children prefer the strengthened ('exactly') reading for bare numerals but not for *some*. On the other hand, Panizza et al. (2009) show that speakers access the dual-bounded ('exactly') readings of numerals more easily in upward-entailing environments than in downward-entailing environments, consistent with saying that the 'exactly' meaning is obtained via implicature and that implicatures are computed more readily in contexts where they would lead to a stronger meaning. Also, Barner and Bachrach (2010) show that children who interpret as exact numerals up to n also have knowledge of $n + 1$, consistent with the scalar implicature story where consideration of higher ranked numerals is needed to obtain the 'exactly' meaning. Conclusion 3: There is support for the 'exactly' meaning of bare numerals being obtained as an implicature, but also for there being a contrast of some sort between the scalar implicatures of bare numerals as opposed to those of other scalar items. Spector (2013) suggests that this contrast could be captured by saying that numerals are intrinsically focused, which would make their scalar alternatives active as a default. Such an assumption, he argues, is supported by evidence showing that bare numerals don't need to be focused in contexts where other scalar items do.

All in all, it seems to me that the scalar implicature account of bare numerals is able to overcome all the objections raised against it so far, and is thus still worth pursuing.

(5)   Alice has three / more than three / less than three / at most three / at least three diamonds.
        ⤳ ¬ Alice has four / *more than four / *less than two / *at most two / *at least four diamonds.
        (Total predicted meaning: She has exactly three / exactly four / exactly two / exactly three / exactly three diamonds.)

Beginning with Krifka (1999), this piece of data has been used to motivate analyses where CMs and SMs either have scalar alternatives that are neutralized (in all or merely some environments), or they don't have scalar alternatives at all. (See §3 for a comprehensive review.)

But is this example sufficient for us to discard a unified scalar implicature story for bare and modifed numerals? In what way exactly do the data deviate from our expectations? Let us review a few more examples.

When embedded under a universal modal such as *require*, BNs, CMs, and SMs can all give rise to the expected direct scalar implicatures, (6). Thus, for example, we can say *Alice is required to have at most three diamonds* to implicate that however she is not required to have at most two (that is, three is an acceptable number).

(6)   Alice is required to have three / more than three / less than three / at most three / at least three diamonds.
        ⤳ ¬ Alice is required to have four / more than four / less than two / at most two / at least four diamonds.

Note, however, that if we calculate the implicature below the embedding operator we again find the pattern from unembedded environments, that is, the predicted scalar implicatures lead to an intuitively plausible total meaning for BNs but not for CMs or SMs.

(7)   Alice is required to have three / more than three / less than three / at most three / at least three diamonds.
        ⤳ Alice is required to have (three ⤳ ¬ four) / (more than three *⤳ ¬ more than four) / (less than three *⤳ ¬ less than two) / (at most three *⤳ ¬ at most two) / (at least three *⤳ ¬ at least four).
        (Total predicted meaning: Alice is required to have exactly three / exactly four / exactly two / exactly three / exactly three diamonds.)

When embedded in the antecedent of a conditional, BNs, CMs, and SM can all trigger indirect[3] scalar implicatures, (8). That is, we can easily take *If Alice has three diamonds, she wins* to implicate . . . *but not if she has two*, or *If Alice has less than three diamonds, she wins* to implicate . . . *but not if she has less than four* (that is, all the situations where the number of diamonds that she has is 0, 1, or 2 are situations where she wins, but not all the situations where it is 0, 1, 2, or *3*).

(8)   If Alice has three / more than three / less than three / at most three / at least three diamonds she wins.
        ⤳ ¬ If Alice has two / more than two / less than four / at most four / at least two diamonds she wins.

(If we calculate the implicature inside the antecedent we get the same mixed pattern as below *require*.)

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[3]Following (Chierchia 2004:59), by *indirect implicatures* I designate implicatures from alternatives based on a lower scalemate. Such alternatives are stronger than the assertion not based on their position on the scale but only by virtue of the numeral showing up in a downward-entailing environment.

However, when the numeral takes scope below negation, none of BNs, CMs, and SMs give rise to the expected indirect implicatures, or else, for example, we would be able to say *Alice doesn't have three diamonds* to mean that she has exactly two (not three and not not two = exactly two), (9). (Note: SMs are known to be degraded under negation. We will discuss this in more detail in §1.4.)

(9)   Alice doesn't have three / more than three / less than three / at most three / at least three diamonds.
        ⤳ ¬ Alice doesn't have *two / *more than two / *less than four / *at most four / *at least two diamonds.
        (Total predicted meaning: She has exactly two / exactly three / exactly three / exactly four / exactly two diamonds.)

(If we calculate the implicature below negation we get the same mixed pattern as below *require*.)

We could continue reviewing further types of examples. However, what we have is already sufficient for us to extract the following two generalizations:

* ⋆ All of BNs, CMs, and SMs sometimes give rise to the expected scalar implicatures and sometimes don't.

* ⋆ All the cases where they don't are cases where the scalar implicature of an utterance of the form *not n, (not) more/less than n*, or *(not) at most/least n* would have led to an 'exactly' reading, (5), (7), (9), and all the cases where the implicatures are as expected are cases where this did not happen.

This leads us to suspect that, if we could prevent the scalar implicatures in those cases from leading to an 'exactly' sort of meaning, we would see implicatures arising as expected. The prediction seems to be confirmed: in a context such as the one below an utterance of *John solved more than five problems* can easily be taken to implicate that he did not solve more than nine. Note that this is the same sort of context as the unembedded context we saw in (5). The difference however is that the implicature together with the assertion does not lead to an 'exactly' meaning, the reason being that the contextually relevant scale is not {*more than three, more than four, …*} (the scale that led to *more than three and not more than four = exactly four* in (5)), but rather {*more than five, more than nine*}.

(10)   (example from Spector 2014:42)
        Context: Grades are attributed on the basis of the number of problems solved. People who solve between one and five problems get a C. People who solve more than five problems but fewer than nine problems get a B, and people who solve 9 problems or more get an A.
        John solved more than five problems. Peter solved more than nine.
        ⤳ ¬ John solved more than nine.

The sort of contextual scale granularity manipulation that helped us get scalar implicatures for *more than n* in (10) above can in fact rescue all our other problem cases, for both bare and modified numerals. For example, in a context that makes salient a scale such as {*not one, not three, …*}, an utterance of *Alice doesn't have three diamonds* can easily trigger the indirect implicature that she does have at least one (not three and not not one = at least one but not three or more). And the same for all the other problem cases.

That CMs and SMs can systematically give rise to scalar implicatures and that these may depend on scale granularity is further supported with experimental evidence by Cummins et al. (2012).

To sum up, we have seen evidence that all of BNs, CMs, and SMs *can* in fact give rise to both direct and indirect implicatures as we would expect if they had scalar alternatives. Additionally we have noticed that the scalar implicatures of *not BN, (not) CM*, or *(not) SM* are blocked only in cases where they would lead to an 'exactly' meaning.

1.3 Ignorance inferences

Beginning with Geurts and Nouwen (2007), a recurrent claim in the literature on numerals is that SMs trigger certain inferences about the epistemic state of the speaker. Coppock and Brochhagen (2013) for example argue that the reason an utterance such as the one below is odd is because the use of the SM suggests that the speaker does not know how many sides a hexagon has.

(11)   #A hexagon has at least five sides.

Similarly, Kennedy (2015) argues that the reason a plane passenger might be alarmed upon hearing the statement below from a flight attendant is because the use of an SM intrinsically indicates ignorance, suggesting crew incompetence.

(12)   This airplane has at least six emergency exits.

Such examples are usually contrasted with minimally different examples as below where the SM is replaced by a BN or a CM and which are argued not to have that effect.

(13)   This airplane has six / more than six emergency exits.

The generalization drawn from such examples is usually that SMs trigger speaker ignorance inferences but BNs and CMs don't, and analyses are developed that would derive ignorance for SMs but not BNs or CMs.

But is this generalization correct? Nouwen (2015) points out that CMs are compatible with ignorance too, and that the difference between CMs and SMs is not that the latter are compatible with ignorance but rather that they *require* it. Indeed, rethinking all of our items in terms of compatibility and requirement, example (14) below shows that BNs are not compatible with an ignorant speaker, but CMs and SMs are:

(14)   a.   I don't know how many diamonds Alice has, #but she has three.
        b.   I don't know how many diamonds Alice has, but she has more than three / less than three / at most three / at least three.

Moreover, example (15) shows that CMs are compatible with exact knowledge but SMs are not.

(15)   (example from Nouwen 2015:244)
        a.   There were exactly 62 mistakes in the manuscript, so that's more than 50.
        b.   There were exactly 62 mistakes in the manuscript, #so that's at least 50.

The correct generalization for plain unembedded contexts thus seems to be a three-way contrast: BNs don't trigger speaker ignorance inferences (marked with ⇝), (16), CMs trigger them optionally, (17), and SMs trigger them obligatorily (Geurts and Nouwen 2007, Nouwen 2010, 2015, Coppock and Brochhagen 2013, Kennedy 2015, Mendia 2015, Spector 2015, a.o.), (18).

(16)   Alice has three (= at least three) diamonds.
        (*⤳ The speaker is not sure whether Alice has three or four or . . . .)

(17)   Alice has more than three / less than three diamonds.
        (⤳ The speaker is not sure whether Alice has four or five or . . . / two or one or . . .)

(18)   Alice has at least three / at most three diamonds.
        *(⤳ The speaker is not sure whether Alice has three or four or . . . / three or two or . . .)

When embedded under a universal modal such as *require*, the ignorance inferences of SMs, just like those of CMs, are optional.

(19)   Alice is required to have three diamonds.
        ⤳̸ The speaker is not sure whether Alice is required to have three or four or . . .

(20)   Alice is required to have more than three / less than three / at most three / at least three diamonds.
        (⤳ The speaker is not sure whether Alice is required to have four or five or . . . / two or one or . . . / three or two or . . . / three or four or . . . .)

And, finally, when embedded under negation, BNs continue not to give rise to ignorance inference and CMs continue to give rise to ignorance inferences, but they seem to be weaker. SMs are known to be degraded under negation (more on this in §1.4), so I will leave them out.

(21)   Alice doesn't have three diamonds.
        ⤳̸ The speaker is not sure whether Alice doesn't have three or four or . . .

(22)   Alice doesn't have more than three / less than three diamonds.
        ⤳ The speaker is sure Alice doesn't have more than three / less than three but not sure how many exactly she does have, i.e., not sure whether she has three or two or . . . / three or four or . . .

To sum up, BNs don't trigger ignorance inferences but CMs and SMs do. In unembedded contexts these inferences are optional for CMs and obligatory for SMs; under a universal operator they are optional for both CMs and SMs; and, finally, under negation, they are weaker for CMs (and presumably absent for SMs).

1.4   Acceptability in downward-entailing environments

BNs and CMs are felicitous under / can take scope below negation, but SMs are degraded on that interpretation (Nilsen 2007, Geurts and Nouwen 2007, Cohen and Krifka 2014, Spector 2015, a.o.; also see Mihoc and Davidson 2017 for experimental evidence). More concretely, a CM-utterance such as *Alice doesn't have more than three diamonds* is easily interpreted as saying that she has three or less, (23), but an SM-utterance such as *Alice doesn't have at least three diamonds* is not easily interpreted as saying that Alice has two or less diamonds, (24).

(23)   Alice doesn't have three / more than three / less than three diamonds.
        → Alice has two or less / three or less / three or more diamonds.   ✓

(24)   Alice doesn't have *at least three / *at most three diamonds.
        → Alice has two or less / four or more diamonds.   ✗

However, SMs are not infelicitous in all types of downward-entailing environments. Just like CMs and BNs, they can be easily interpreted in the antecedent of a conditional or the restriction of a universal (Nilsen 2007, Geurts and Nouwen 2007, Cohen and Krifka 2014, Spector 2015; also see Mihoc and Davidson 2017 for experimental evidence):

(25)   If Alice has three / more than three / less than three / at least three / at most three diamonds, she wins.

(26)   Everyone who has three / more than three / less than three / at least three / at most three diamonds wins.

To sum up, BNs and CMs are felicitous under negation but SMs aren't. SMs are however just as good as BNs or CMs in the antecedent of a conditional or the restriction of a universal.

1.5   Summary: Basic desiderata for a theory of numerals

To sum up, a theory of BNs, CMs, and SMs should capture all their patterns of similarity and difference with respect to:

- ⋆ entailments

- ⋆ scalar implicatures

- ⋆ ignorance inferences

- ⋆ acceptability in downward-entailing environments

As we will see in detail in our literature review in §3, many of the existing theories struggle with the upper-bounding entailments of *less than* and *at most* (e.g., Krifka 1999 resorts to falsity conditions, Hackl 2000 to a maximality operator in the meaning of *less than* and to negation in the meaning of *at most*, etc.); (to my knowledge) none manage to capture all the scalar implicature patterns, typically failing to account for both the indirect scalar implicatures of all of bare and modified numerals and the direct scalar implicatures in unembedded contexts of CMs and SMs that we saw in (10) (e.g., in various ways, Krifka 1999, Fox and Hackl 2006, Mayr 2013, Kennedy 2015, or Spector 2015 all design their theories such that no scalar implicatures can arise in an unembedded context for either CMs, or for SMs, or for both); none manage to capture all the ignorance inference patterns, failing to explain either the obligativity of these inferences for SMs in certain contexts but not others, or their availability for CMs also, or both (e.g., Kennedy 2015 sets up the alternative set such that SMs are able to get ignorance implicatures but not CMs); and most fail to capture the acceptability in downward-entailing environments facts (e.g., none of Coppock and Brochhagen 2013, Mayr 2013, or Kennedy 2015 predict any contrast between CMs and SMs under negation).

In the next section I will propose a unified theory of bare and modified numerals that captures all these patterns in full.

2   A unified theory of bare and modified numerals

In this section I propose a unified, extent- and alternative-based, theory of bare, comparative-modified, and superlative-modified numerals and show how it accounts for all the data presented before, and

more. The structure of the section is as follows: In §2.1 I introduce the new theory. In §2.2 I show how it captures all the empirical patterns discussed before from the interaction of the morphological pieces of BNs, CMs, and SMs, on the one hand, and general pragmatic (= implicature-calculation) mechanisms, on the other. In §2.3 I discuss how it bears on additional issues. In §2.4 I sum up.

2.1 Theory

n P Q, *more/less than n P Q*, and *at most/least n P Q* obviously share many morphological pieces. All of them share a numeral; the modified numerals share *much/little*; comparative-modified numerals share a comparative meaning; and superlative-modified numerals share a superlative meaning. I will argue that the key to a unified account of bare and modified numerals that captures both their similarities and their differences lies with the proper understanding of these pieces and of their contribution to the whole.

In §2.1.1 I break down our bare and modified numerals into their component pieces and then put them back together compositionally, extracting from them both truth conditions and an important presupposition. In §2.1.2 I show how from these truth conditions and presupposition one can naturally define scalar and subdomain alternatives as well as their functioning. In §2.1.3 I introduce the implicature calculation system I will be using. In §2.1.4 I summarize everything in a little cheatsheet.

2.1.1 Morphological breakdown: truth conditions and presupposition

Taking the overt morphology at face value, I propose that *n P Q*, *more/less than n P Q*, *at most/least n P Q* must each be decomposed into a subset of the following pieces: a numeral *n*, a positive extent indicator *much* / negative extent indicator *little*, a comparative morpheme [comp] / superlative morpheme [sup], a predicate with a nominal meaning *P*, and a predicate with a verbal meaning *Q*, as shown in (27).

(27)

n P Q	(n)(P)(Q)
more/less than n P Q	[comp](much/little)(n)(P)(Q)
at most/least n P Q	[sup](much/little)(n)(P)(Q)

I will assume that the numeral in bare and modified numerals is just a degree, (28-a). If needed (e.g., in type mismatch situations), this degree may undergo typeshifting into a predicate via something like Buccola and Spector (2016)'s isCard typeshifter, (28-b).[4]

(28) a. $[\![n]\!] = n$
      b. $[\![isCard]\!] (n) = \lambda x \, . \, |x| = n$

The truth conditions for *n P Q* are then obtained as follows. *n* and *P* form a constituent, so they must be put together first. *n* is a degree, so in order to combine with *P* it must first undergo typeshifting into a predicate (the meaning in (28-b)), after which the two can come together via predicate modification, yielding another predicate. The (*n P*) predicate is then closed by a silent existential quantifier $\varnothing_\exists$, which turns the DP into a generalized quantifier that can in turn be saturated with Q.

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[4]Note that the opposite is also conceivable, namely, to take the predicative meaning in (28-b) as basic and then derive from it the degree meaning.

(29) *n P Q*
(∃ (n P))(Q)
$= 1$ iff $\exists x[|x| = n \land P(x) \land Q(x)]$

More concretely,

(30) *Three students smiled*
(∃ (three students))(smiled)
$= 1$ iff $\exists x[|x| = 3 \land \text{students}(x) \land \text{smiled}(x)]$

This is essentially an 'at least' semantics for bare numerals (as in Horn 1972 or Krifka 1999): By asserting the existence of a plurality with three atoms, this in principle leaves open that the total number of x's that satisfy both P and Q is either three or larger than three.

Moving on to the next pair of pieces in our morphological breakdown, I propose that the *much/little* that we see in *more/less than n P Q* and *at least/most n P Q* are extent indicators mapping a degree n to its positive or negative extent. As in Kennedy (1997:251-2), given a degree n belonging to a scale S, the positive / negative extent of n on S is defined as a non-empty, convex subset of S (i.e., an interval on S) denoting the set of degrees less than or equal to n, (31-a) / the set of degrees greater than or equal to n, (31-b).[5]

(31) a. the positive extent of n: b. the negative extent of n:
$\llbracket \text{much} \rrbracket (n) = \lambda d . d \leq n,$ $\llbracket \text{little} \rrbracket (n) = \lambda d . d \geq n,$
where $n, d \in S$ where $n, d \in S$
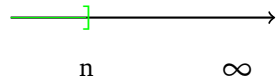
We are now ready to tackle the meaning of comparative- and superlative-modified numerals.

I propose that the comparative morpheme [comp] in CMs is a function that takes in $\llbracket \text{much/little} \rrbracket$, n, P, and Q, and yields true iff the cardinality of the intersection of P and Q is a number in the complement of $\llbracket \text{much/little} \rrbracket (n)$, that is, in the complement of the positive/negative extent of n. More intuitively, the idea is that $|P \cap Q|$ is 'much' or 'little' to a degree to which n is not.

(32) *more/less than n P Q*
$\llbracket [\text{comp}] \rrbracket (\llbracket \text{much/little} \rrbracket)(n)(P)(Q)$
$= 1$ iff $|P \cap Q| \in \overline{\llbracket \text{much/little} \rrbracket (n)}$

More concretely, (assuming a scale with granularity 1,)

(33) *More than three students smiled*
$\llbracket [\text{comp}] \rrbracket (\llbracket \text{much} \rrbracket)(\llbracket \text{three} \rrbracket)(\llbracket \text{students} \rrbracket)(\llbracket \text{smiled} \rrbracket)$
$= 1$ iff $|\llbracket \text{students} \rrbracket \cap \llbracket \text{smiled} \rrbracket| \in \overline{\llbracket \text{much} \rrbracket (3)}$
$= 1$ iff $|\llbracket \text{students} \rrbracket \cap \llbracket \text{smiled} \rrbracket| \in \overline{\{0, 1, 2, 3\}}$
$= 1$ iff $|\llbracket \text{students} \rrbracket \cap \llbracket \text{smiled} \rrbracket| \in \{4, 5, \ldots\}$

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[5]This makes them similar to adjectives in the sense of Seuren (1984)/Kennedy (1997), except *many/little* are type $\langle d, \langle d, t \rangle \rangle$ instead of $\langle e, \langle d, t \rangle \rangle$, and the scale is a cardinality scale rather than a scale measuring some other property (e.g., height).

(34) *Less than three students smiled*
      $[\![\text{comp}]\!]\,([\![\text{little}]\!])([\![\text{three}]\!])([\![\text{students}]\!])([\![\text{smiled}]\!])$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in \overline{[\![\text{little}]\!]\,(3)}$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in \overline{\{3,4,5,\dots\}}$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in \{0,1,2\}$

Finally, the superlative morpheme [sup] in SM is a function that takes in $[\![\text{much/little}]\!]$, n, P, Q, and yields true iff the cardinality of the intersection of P and Q is a number in $[\![\text{much/little}]\!]\,(n)$, that is, in the positive/negative extent of n. More intuitively, the idea is that $|P \cap Q|$ is at most as 'much' or as 'little' as n is.

(35) *at most/least n P Q*
      $[\![\text{sup}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$
      $= 1$ iff $|P \cap Q| \in [\![\text{much/little}]\!]\,(n)$

More concretely, (assuming a scale with granularity 1,)

(36) *At most three students smiled*
      $[\![\text{sup}]\!]\,([\![\text{much}]\!])([\![\text{three}]\!])([\![\text{students}]\!])([\![\text{smiled}]\!])$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in [\![\text{much}]\!]\,(3)$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in \{0,1,2,3\}$

(37) *At least three students smiled*
      $[\![\text{sup}]\!]\,([\![\text{little}]\!])([\![\text{three}]\!])([\![\text{students}]\!])([\![\text{smiled}]\!])$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in [\![\text{little}]\!]\,(3)$
      $= 1$ iff $|\,[\![\text{students}]\!] \cap [\![\text{smiled}]\!]\,| \in \{3,4,\dots\}$

In addition to producing these truth conditions, the superlative morpheme also contributes an important presupposition. As we know from the literature on superlative-modified adjectives (see Heim 1999, Gajewski 2010, a.o.), this is a presupposition about the minimum size of the comparison class which requires that it must have at least two (possibly even three) elements. For example, in a superlative-modified adjective such as *tallest* the comparison class is a set of individuals, and the presupposition amounts to a requirement that there must be at least two (possibly even three) individuals in the domain of entities (Gajewski 2010). I will propose that in SMs the relevant comparison class is a set of degrees, namely the set of degrees in the positive/negative extent of n. The presupposition of [sup] for SMs thus amounts to a requirement that the domain of d, namely, $[\![\text{much/little}]\!]\,(n)$, must contain at least two degrees:

(38) $[\![\text{sup}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$ defined iff $|\,[\![\text{much/little}]\!]\,(n)\,| \geq 2$

As we will see in §2.1.2, this presupposition can help us explain the special status of the subdomain alternatives of SMs.

We now have our basic meanings for the numeral, *much/little*, [comp], and [sup], and thereby also the truth conditions for BNs, CMs, and SMs, and a presupposition for SMs stemming from the meaning of [sup]. These meanings not only give us the natural compositional semantics for BNs, CMs, and SMs that we saw above, but also, as we will see next, a semantics from which their alternatives – including their type (scalar or subdomain) and status status (optional or obligatory) – can be naturally read off.

2.1.2 Scalar and subdomain alternatives

The truth conditions above give us a natural way to articulate scalar alternatives for all of BNs, CMs, and SMs. These can be uniformly derived by replacing n in the truth conditions with another numeral alternative from a scale S with a contextually-determined granularity. Aside from the caveat that the scale may not always be the scale of natural numbers (e.g., the scale may have a different granularity), these alternatives are essentially the same as outlined for the traditional story in (4). More concretely, for a traditional scale of positive numbers with granularity 1, the scalar alternatives of CMs and SMs would be as follows:

(39) ScalAlts(*three P Q*)
$= \text{ScalAlts}(\exists x[|x| = 3 \wedge P(x) \wedge Q(x)])$
$= \{\ldots, \exists x[|x| = 2 \wedge P(x) \wedge Q(x)], \exists x[|x| = 4 \wedge P(x) \wedge Q(x)], \ldots\}$
$= \{\ldots, two\ P\ Q, four\ P\ Q, \ldots\}$

(40) ScalAlts(*more/less than three P Q*)
$= \text{ScalAlts}(|P \cap Q| \in \overline{[\![\text{much/little}]\!]}\,(3))$
$= \{\ldots, |P \cap Q| \in \overline{[\![\text{much/little}]\!]}\,(2), |P \cap Q| \in \overline{[\![\text{much/little}]\!]}\,(4), \ldots\}$
$= \{\ldots, more/less\ than\ two\ P\ Q, more/less\ than\ four\ P\ Q, \ldots\}$

(41) ScalAlts(*at most/least three P Q*)
$= \text{ScalAlts}(|P \cap Q| \in [\![\text{much/little}]\!]\,(3))$
$= \{\ldots, |P \cap Q| \in [\![\text{much/little}]\!]\,(2), |P \cap Q| \in [\![\text{much/little}]\!]\,(4), \ldots\}$
$= \{\ldots, at\ most/least\ two\ P\ Q, at\ most/least\ four\ P\ Q, \ldots\}$

The truth conditions above also give us a natural way to articulate domain-related alternatives for CMs and SMs but not BNs. Note that, due to the presence of *much/little* in their meaning, the truth conditions of CMs and SMs are obtained by direct reference to a set of degrees. More specifically, for CMs the value of $|P \cap Q|$ is given by the set of degrees denoted by complement of the positive/negative extent of n ($\overline{[\![\text{much/little}]\!]}\,(n)$), and for SMs it is given by the set of degrees denoted by the positive/negative extent of n ($[\![\text{much/little}]\!]\,(n)$). Subdomain alternatives can then be naturally and uniformly derived for CMs and SMs by replacing this set of degrees with its subsets. Intuitively this amounts to alternatives where the value of $|P \cap Q|$ is given by domains smaller than the domain in the assertion. More concretely, (assuming again a traditional scale of positive numbers with granularity 1 and therefore only including in our domains degrees 0 or greater,) the subdomain alternatives of CMs and SMs are as follows:

(42) SubDomAlts(*more/less than three P Q*)
$= \text{SubDomAlts}(|P \cap Q| \in \overline{[\![\text{much/little}]\!]}\,(3))$
$= \text{SubDomAlts}(|P \cap Q| \in \{4, 5, \ldots\}/\{0, 1, 2\})$
$= \{|P \cap Q| \in \{4\}, |P \cap Q| \in \{4, 7\}, \ldots\} \,/\, \{|P \cap Q| \in \{0\}, |P \cap Q| \in \{0, 1\}, \ldots\}$

(43) SubDomAlts(*at most/least three P Q*)
$= \text{SubDomAlts}(|P \cap Q| \in [\![\text{much/little}]\!]\,(3))$
$= \text{SubDomAlts}(|P \cap Q| \in \{0, 1, 2, 3\}/\{3, 4, \ldots\})$
$= \{|P \cap Q| \in \{0\}, |P \cap Q| \in \{1, 3\}, \ldots\} \,/\, \{|P \cap Q| \in \{3\}, |P \cap Q| \in \{4, 8\}, \ldots\}$

In contrast, the meaning of BNs does not rely in its composition on any set of degrees, so no such subdomain alternatives are possible.

While the subdomain alternatives of CMs and SMs are similar in the sense that they are derived

in the same way, there are also reasons to believe that they might be different in status. If, as put forth in (38), SMs presuppose, because of the superlative morpheme [sup], that the domain of $|P \cap Q|$, that is, $[\![much/little]\!](n)$, must contain at least two elements, this also means that we always have two subdomain alternatives (based on the singleton sets corresponding to these two elements) active by presupposition. Once active, an alternative has to be used, so I argue that this results in a requirement that an implicature based on the subdomain alternatives of SMs must always be computed. (This parallels a similar proposal for the domain alternatives of SMs by Spector 2015. Spector however does not give any explanation as to why the domain alternatives of SMs should be obligatory. On the current account this comes out of the domain-size presupposition of the superlative morpheme. See §3.12 for further comparison of the present account to Spector's account.)

As I will show, the truth conditions and alternatives above can help us make sense of all the meanings and distribution of BNs, CMs, and SMs. But before we can see that we need to set up our implicature calculation system.

2.1.3 Implicature calculation system

First, we have to decide whether we should use the traditional Gricean theory of implicatures (Grice 1975), on which implicatures are a discourse-level phenomenon, or the more recent grammatical theory of scalar implicatures (Chierchia 2004, Fox 2007, Chierchia et al. 2012, Chierchia 2013), on which Gricean reasoning is replaced with a grammatical operator able to occur at embedded levels also. I will choose the latter, namely, the grammatical theory of scalar implicatures. As we will see, this choice is not only a matter of convenience but it crucially allows us to derive embedded scalar implicatures (which, given our assumptions about BNs, are needed to capture the 'exactly' meaning of BNs in embedded environments).

Second, we have to choose between the two main variants of the grammatical theory of implicatures, namely, the contradiction-free system, best summarized in Chierchia et al. (2012), or the contradiction-based system, best summarized in Chierchia (2013). I will again choose the latter. This choice is merely a matter of convenience: although I believe that the two systems may ultimately yield the same results,[6] the contradiction-based system allows for greater ease of exposition.[7]

To recap, for implicature calculation I will be using the contradiction-based variant of the grammatical theory of implicatures (Chierchia 2013). More specifically, we will need the following features:

A silent exhaustivity operator O Given a sentence ϕ and a set ALT of alternatives to ϕ, $O_{ALT}(\phi)$ asserts ϕ (also called the *prejacent*) and furthermore says that all the alternatives in ALT that are true are already entailed by ϕ – that is, it says that all of the non-entailed alternatives of ϕ are false. If all the

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[6] The two systems seem to differ in how they handle parses where exhaustication does not lead to strengthening (i.e., leads to a weaker meaning or is vacuous). The contradiction-based system defines two exhaustivity operators – a plain exhaustivity operator, O, which tolerates exhaustification parses that lead to weakening, and an exhaustivity operator with a requirement for proper strengthening, O^{PS}, which does not. On the other hand, the contradiction-free system uses a blanket economy condition ruling out exhaustification parses where the result does not lead to strengthening. In what follows all the results that we get in the contradiction-based system with $O^{PS}_{SubDomAlts}$ can be replicated in the contradiction-free system with exhaustification + the economy condition.

[7] Aside from incarnating Gricean reasoning in the form of an operator, the contradiction-based system derives implicatures in the same way through the negation of the non-entailed alternatives. On the other hand, the contradiction-free system does not negate all the non-entailed alternatives but merely those among them that are *Innocently Excludable* – that is, roughly speaking, neither symmetric, nor entailed.

alternatives are already entailed, $O_{ALT}(\phi)$ simply returns the assertion (Chierchia 2013:274) (vacuous exhaustification).

(44) $\llbracket O_{ALT}(\phi) \rrbracket^{g,w} = \llbracket \phi \rrbracket^{g,w} \wedge \forall p \in \llbracket \phi \rrbracket^{ALT} [p \to \lambda w' . \llbracket \phi \rrbracket^{g,w'} \subseteq p]$ (Chierchia 2013:139)

I will assume that scalar alternatives are exhaustified this way.

A silent exhaustivity operator O^{PS} 'O with proper strengthening'   Given a sentence ϕ and a set ALT of alternatives to ϕ, $O^{PS}_{ALT}(\phi)$ (1) is defined iff the *strong* exhaustification of ϕ leads to a meaning strictly stronger than ϕ (i.e., only if exhaustification is not vacuous), and, whenever defined, (2) yields the strong exhaustification of ϕ, (45) – where the strong exhaustification of ϕ, $O^S(\phi)$, is defined as exhaustification that looks at the *presupposition-enriched* content of ϕ, $\pi(\phi)$ and the *presupposition-enriched* content of the alternatives, $\pi(p)$, (45-a), where in turn the presupposition-enriched content of a proposition p, $\pi(q)$, is defined as the sum of the assertive component of q, $^{\alpha}q$, and of the presuppositional content of q, $^{\pi}q$, (45-a-i):

(45) $O^{PS}_{ALT}(\phi)$ is defined iff $O^S_{ALT}(\phi) \subset \phi$. Whenever defined, $O^{PS}_{ALT}(\phi) = O^S_{ALT}(\phi)$,
        where   (Chierchia 2013:274)[8]
        a. $O^S_{ALT}(\phi_w) = \phi_w \wedge \forall p \in ALT \, [\pi(p)_w \to \pi(\lambda w . \phi_w) \subseteq \pi(p)]$,   (Chierchia 2013:220)
             where
             (i) $\pi(q) = {}^{\alpha}q \wedge {}^{\phi}(q)$.   (Chierchia 2013:219)

More intuitively, O^{PS} can be regarded as a moodier version of O whose use is felicitous only if it would lead to a properly stronger meaning (if exhaustification is not vacuous), and which is happy to allow the presuppositional content of the prejacent to contribute to the satisfaction of this felicity condition.

I will assume that CMs and SMs are lexically specified[9] for exhaustification via $O^{PS}_{SubDomAlts}$ – that is, exhaustification relative to their subdomain alternatives must take into account both assertive and presuppositional content, and it must lead to a stronger meaning. Moreover, if, as proposed at the end of §2.1.1, the subdomain alternatives of SMs are always active, this means that SMs must *always* be in the scope of $O^{S,PS}$ – that is, implicatures coming from this sort of exhaustification are obligatory for SMs. As we will see in §2.2.3 and §2.2.4, all these assumptions will help us derive the ignorance and acceptability in downward-entailing environments patterns of CMs and SMs.[10]

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[8] Chierchia (2013:274)'s original definition of O^{PS} doesn't specify that the type of exhaustification involved in O^{PS} is exhaustification in the strong sense, that is, O^S. However, those items whose behavior can be captured with O^{PS} always seem to crucially require exhaustification in the strong sense, so I have decided to absorb O^S into the meaning of O^{PS}. If it turns out that there are items whose behavior can be captured with O^{PS} but crucially do not need exhaustification in the strong sense, we may want to further parametrize O^{PS} for strong vs. weak exhaustification, $O^{PS,\,S}$ vs. $O^{PS,\,W}$.

[9] Chierchia (2013) suggests that various items may be *lexically specified* for whether they can undergo plain exhaustification, or must undergo exhaustification with proper strengthening, showing that from this we can derive their distribution.

[10] The idea that the ignorance implicatures of SMs must be derived from some sort of domain alternatives has been proposed before, by Büring (2008), Kennedy (2015), or Spector (2015), a.o. The idea that we can get the obligatoriness of ignorance for SMs in certain contexts as well as their infelicity in DE environments by saying that their domain alternatives are obligatory and must lead to strengthening has also already been proposed (in a slightly different form but to the same effect) by Spector (2015). However, the present account improves on those accounts by extending the account to CMs and moreover deriving the form, type, and status of the subdomain alternatives of CMs and SMs in a more principled way. For a more concrete discussion of how this account improves on Kennedy's account and Spector's account see §3.11 and §3.12, respectively.

A last resort, silent, matrix-level, universal epistemic/doxastic modal ⊡ (my notation) Whether (neo-)Gricean or grammatical, implicature calculation systems typically also include reference to some sort of a silent, matrix-level, universal epistemic modal. This modal is usually written as Bel_S 'the speaker believes that …', or as K, or simply as a universal modal $\Box$. There is also a debate whether it should be regarded as a grammatical operator (as argued by Meyer 2013), and also whether it should prefix every assertive sentence or instead be used merely as a rescue mechanism (as used by Kratzer and Shimoyama 2002 or Chierchia 2013:256). In what follows I will write it as ⊡ ('box-dot'; the dot is there simply to distinguish it typographically from an overt universal modal) and conceptualize it with Kratzer and Shimoyama (2002) or Chierchia (2013:256) as a *last resort*, silent, matrix-level, epistemic modal that can be inserted between the exhaustivity operator and its prejacent in order to rescue an exhaustification parse that would otherwise crash. This choice however is not in any way crucial and is merely for ease of exposition.[11]

These are all our assumptions about implicature calculation going forward.

2.1.4 Summary of assumptions (theory cheatsheet)

To sum up, the truth conditions and alternatives of BNs, CMs, and SMs are as follows:

(46) **Bare numerals**
n P Q
$(\exists\,(n\,P))(Q)$

    a. $\exists x[|x| = n \wedge P(x) \wedge Q(x)]$ (truth conditions)
    b. $\exists x[|x| = m \wedge P(x) \wedge Q(x)],\ m \in S$ (scalar alternatives)
    c.   N/A   (no subdomain alternatives)

(47) **Comparative-modified numerals**
more/less than n P Q
$[\![\text{comp}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$

    a. $|P \cap Q| \in \overline{[\![\text{much/little}]\!]\,(n)}$ (truth conditions)
    b. $|P \cap Q| \in \overline{[\![\text{much/little}]\!]\,(m)},\ m \in S$ (scalar alternatives)
    c. $|P \cap Q| \in A,\ A \subseteq \overline{[\![\text{much/little}]\!]\,(n)}$ (subdomain alternatives)

(48) **Superlative-modified numerals**
at most/least n P Q
$[\![\text{sup}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$

    a. $|P \cap Q| \in [\![\text{much/little}]\!]\,(n)$
    b. $|P \cap Q| \in [\![\text{much/little}]\!]\,(m),\ m \in S$ (scalar alternatives)

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[11] Although I believe it is a choice that could prove advantageous in certain cases as it could help avoid certain superfluous parses. For example, the traditional neo-Gricean account of BNs parses *three P Q* with this modal at the top, that is, as $Bel_S\,[\![\text{three P Q}]\!]$; this generates an undesired implicature of the form $\neg Bel_S\,[\![\text{four P Q}]\!]$ (essentially, an ignorance implicature coming from a scalar alternative), and additional mechanisms (e.g., the Opinionated Speaker Assumption, cf. Fox 2007, from Sauerland 2005) are needed to strengthen it into the desired scalar implicature $Bel_S\neg\,[\![\text{four P Q}]\!]$. On the other hand, if we conceptualize this modal as a last resort mechanism, the input to the exhaustivity operator would not include the modal (because exhaustification can proceed perfectly well without it), so we would not generate the unattested ignorance implicature in the first place. See §2.2 for concrete examples. Moreover, in addition to the fact that it would help prevent overgeneration, conceptualizing this modal as a covert rescue mechanism would also help capture a parallelism with cases where an overt modal seems to have the same effect (Chierchia 2013).

c. $|P \cap Q| \in A,\ A \subseteq \llbracket \text{much/little} \rrbracket (n)$ (subdomain alternatives, active by presup.)

And our assumptions about how the alternatives will be used are as follows:

The scalar alternatives of BNs, CMs, and SMs are factored in via O.

The subdomain alternatives of CMs and SMs are factored in via O^{PS} (exhaustification with proper strengthening). For SMs this is obligatory (in other words, logical forms where an SM is not in the scope of $O^{PS}_{SubDomAlts}$ are banned). When exhaustification fails (either because it leads to contradiction or, in the case of O^{PS}, because it does not lead to strengthening), a silent epistemic modal $\Box$ may be inserted at matrix level in between the exhaustivity operator and the prejacent as a last resort rescue mechanism.

2.2 Capturing the basic patterns

Let's see step by step how our theoretical setup in §2.1 helps us derives the empirical patterns under discussion.

2.2.1 Entailments

Recall that *three P Q, more than three P* , and *at least three P Q* all entail a lower bound, and *less than three P Q* and *at most three P Q* all entail an upper bound.

Our new meanings for *3 P Q, more than 3 P Q*, and *at least 3 P Q* all place the value of $|P \cap Q|$ in a lower-bounded interval. In consequence they all entail a lower bound (and are implicitly incompatible with values below that bound).

(49)   3 P Q:
$\exists x[|x| = 3 \wedge P(x) \wedge Q(x)] \Rightarrow |P \cap Q| \geq 3$ (lower bound)

(50)   more than 3 P Q:
$|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} (3) \Longleftrightarrow |P \cap Q| \in \{4, 5, \dots\}$ (lower bound)

(51)   at least 3 P Q:
$|P \cap Q| \in \llbracket \text{little} \rrbracket (3) \Longleftrightarrow |P \cap Q| \in \{3, 4, \dots\}$ (lower bound)

Similarly, *less than 3 P Q* and *at most 3 P Q* both place the value of $|P \cap Q|$ in an upper-bounded interval. In consequence they entail an upper-bound (and are implicitly incompatible with values above that bound).

(52)   less than 3 P Q:
$|P \cap Q| \in \overline{\llbracket \text{little} \rrbracket} (3) \Longleftrightarrow |P \cap Q| \in \{\dots, 0, 1, 2\}$ (upper bound)

(53)   at most 3 P Q:
$|P \cap Q| \in \llbracket \text{much} \rrbracket (3) \Longleftrightarrow |P \cap Q| \in \{\dots, 0, 1, 2, 3\}$ (upper bound)

Our truth conditions thus straightforwardly give us the bounding entailments in each case.

2.2.2 Scalar implicatures

In keeping with the original scalar implicature view of numerals, all of BNs, CMs, and SMs are predicted to get scalar implicatures by the negation of their stronger scalar alternatives. Just like in the classical story, this yields some good results but also some bad results.

First, in unembedded environments we get the right results for BNs but not for CMs or SMs:

(54) $O_{ScalAlts}$ *(Alice has three diamonds)*
$O_{ScalAlts}$ $(\exists x[|x| = 3 \land P(x) \land Q(x)])$
$= \exists x[|x| = 3 \land P(x) \land Q(x)] \land \neg \exists x[|x| = 4 \land P(x) \land Q(x)]$
'Alice has exactly three diamonds.' ✓

(55) $O_{ScalAlts}$ *(Alice has $\overline{\text{more than}}$ three diamonds)*
$O_{ScalAlts}$ $(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3))$
$= |P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3) \land \neg |P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(4)$
'Alice has exactly four diamonds.' ✗

(56) $O_{ScalAlts}$ *(Alice has at least three diamonds)*
$O_{ScalAlts}$ $(|P \cap Q| \in \llbracket \text{much} \rrbracket(3))$
$= |P \cap Q| \in \llbracket \text{much} \rrbracket(3) \land \neg |P \cap Q| \in \llbracket \text{much} \rrbracket(4)$
'Alice has exactly three diamonds.' ✗

When the numeral is embedded under a universal operator and we exhaustify above the operator, we get the correct direct scalar implicatures for all:

(57) $O_{ScalAlts}$ *(Alice is required to have three diamonds)*
$O_{ScalAlts}$ $(\Box(\exists x[|x| = 3 \land P(x) \land Q(x)]))$
$= \Box(\exists x[|x| = 3 \land P(x) \land Q(x)]) \land \neg \Box(\exists x[|x| = 4 \land P(x) \land Q(x)])$
'Alice is required to have at least three diamonds but she is not required to have at least four.' ✓

(58) $O_{ScalAlts}$ *(Alice is required to have $\overline{\text{more than}}$ three diamonds)*
$O_{ScalAlts}$ $(\Box(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3)))$
$= \Box(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3)) \land \neg \Box(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(4))$
'Alice is required to have more than three diamonds but she is not required to have more than four.' ✓

(59) $O_{ScalAlts}$ *(Alice is required to have at least three diamonds)*
$O_{ScalAlts}$ $(\Box(|P \cap Q| \in \llbracket \text{much} \rrbracket(3)))$
$= \Box(|P \cap Q| \in \llbracket \text{much} \rrbracket(3)) \land \neg \Box(|P \cap Q| \in \llbracket \text{much} \rrbracket(4))$
'Alice is required to have at least three diamonds but she is not required to have at least four.' ✓

When the numeral is embedded under a universal operator and we exhaustify *below* the operator, we get the right results for BNs but the wrong results for CMs or SMs:

(60) *Alice is required* $O_{ScalAlts}$ *(to have three diamonds)*
$\Box(\exists x[|x| = 3 \land P(x) \land Q(x)] \land \neg \exists x[|x| = 4 \land P(x) \land Q(x)])$
'Alice is required to have exactly three diamonds.' ✓

(61) *Alice is required* $O_{ScalAlts}$ *(to have $\overline{\text{more than}}$ three diamonds)*
$\Box(O_{ScalAlts}$ $(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3)))$
$= \Box(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(3) \land \neg |P \cap Q| \in \overline{\llbracket \text{much} \rrbracket}(4))$
'Alice is required to have exactly four diamonds.' ✗

(62) *Alice is required* $O_{ScalAlts}$ *(to have at least three diamonds)*
$\Box(O_{ScalAlts}$ $(|P \cap Q| \in \llbracket \text{little} \rrbracket(3)))$

$= \Box(|P \cap Q| \in \llbracket \text{little} \rrbracket \, (3) \wedge \neg |P \cap Q| \in \llbracket \text{little} \rrbracket \, (4))$
'Alice is required to have exactly three diamonds.' ✗

When the numeral is embedded in the antecedent of a conditional and we exhaustify *above* the operator, we get the right indirect scalar implicatures for all:

(63) $O_{ScalAlts}$ *(If Alice has three diamonds she wins)*
$O_{ScalAlts}$ (if$[\exists x[|x| = 3 \wedge P(x) \wedge Q(x)]][\dots])$
$= \text{if}[\exists x[|x| = 3 \wedge P(x) \wedge Q(x)]][\dots] \wedge \neg \text{if}[\exists x[|x| = 2 \wedge P(x) \wedge Q(x)]][\dots]$
'All the situations where Alice has three diamonds are situations where she wins, but not all the situations where she has two.' ✓

(64) $O_{ScalAlts}$ *(If Alice has more than three diamonds she wins)*
$O_{ScalAlts}$ (if$[|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (3)][\dots])$
$= \text{if}[|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (3)][\dots] \wedge \neg \text{if}[|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (2)][\dots]$
'All the situations where Alice has more than three diamonds are situations where she wins, but not all the situations where she has more than two.' ✓

(65) $O_{ScalAlts}$ *(If Alice has at least three diamonds she wins)*
$O_{ScalAlts}$ (if$[|P \cap Q| \in \llbracket \text{much} \rrbracket \, (3)][\dots])$
$= \text{if}[|P \cap Q| \in \llbracket \text{much} \rrbracket \, (3)][\dots] \wedge \neg \text{if}[|P \cap Q| \in \llbracket \text{much} \rrbracket \, (2)][\dots]$
'All the situations where Alice has at least three diamonds are situations where she wins, but not all the situations where she has at least two.' ✓

(When the numeral is embedded in the antecedent of a conditional and we exhaustify *below* the operator, we get the same mixed results as in the unembedded case or in the case where we were exhaustifying below a universal operator.)

In general, thus, given a standard scale with granularity 1, our scalar alternatives yield mixed results, producing both attested implicatures but also some that are not attested, more specifically, some leading to an 'exactly' meaning for CMs and SMs.

There is however a principled way to rule out the bad results. Recall, for example, that an utterance of *Alice has at least three diamonds* triggers ignorance implicatures to the effect that the speaker is ignorant about exactly how many diamonds Alice has. This would naturally clash with a scalar implicature to the effect that the speaker knows that Alice has exactly three diamonds. I thus propose that the missing scalar implicatures of CMs and SMs are due to a clash between the scalar implicatures and the ignorance implicatures (or, more generally, implicatures from subdomain alternatives) of CMs and SMs. We will discuss this clash, as well as the reason why it is settled against the offending scalar implicature, in the next section. The problem of generated unattested scalar implicatures for CMs and SMs thus goes away.

On the other hand, allowing CMs and SMs to have scalar implicatures in the general case helps us capture the fact that they do give rise to implicatures in contexts with scales of other granularities, as shown below:

(66) $O_{ScalAlts}$ *(Alice has more than three diamonds)*
$O_{ScalAlts}$ $(|P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (3))$
$= |P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (3) \wedge \neg |P \cap Q| \in \overline{\llbracket \text{much} \rrbracket} \, (6)$
'Alice has more than three diamonds but not more than six.' ✓

(67) $O_{ScalAlts}$ *(Alice has at least three diamonds)*
$O_{ScalAlts}$ $(|P \cap Q| \in [\![much]\!](3))$
$= |P \cap Q| \in [\![much]\!](3) \wedge \neg |P \cap Q| \in [\![much]\!](7)$
'Alice has at least three diamonds but not at least seven.'   ✓

But what happens in the case of negation? When the numeral is embedded under negation we get bad results not just for CMs and SMs but for BNs also.

(68) $O_{ScalAlts}$ *(Alice doesn't have three diamonds)*
$O_{ScalAlts}$ $(\neg(\exists x[|x| = 3 \wedge P(x) \wedge Q(x)]))$
$= \neg(\exists x[|x| = 3 \wedge P(x) \wedge Q(x)]) \wedge \neg\neg(\exists x[|x| = 2 \wedge P(x) \wedge Q(x)])$
'Alice has exactly two diamonds.'   ✗

(69) $O_{ScalAlts}$ *(Alice doesn't have more than three diamonds)*
$O_{ScalAlts}$ $(\neg(|P \cap Q| \in \overline{[\![much]\!](3)}))$
$= \neg(|P \cap Q| \in \overline{[\![much]\!](3)}) \wedge \neg\neg(|P \cap Q| \in \overline{[\![much]\!](2)})$
'Alice has exactly three diamonds.'   ✗

(70) $O_{ScalAlts}$ *(Alice doesn't have at least three diamonds)*
$O_{ScalAlts}$ $(\neg(|P \cap Q| \in [\![much]\!](3)))$
$= \neg(|P \cap Q| \in [\![much]\!](3)) \wedge \neg\neg(|P \cap Q| \in [\![much]\!](2))$
'Alice has exactly two diamonds.'   ✗

(Although when the numeral is embedded under negation and we exhaustify *below* negation we get the same mixed results as in the unembedded case or in the case where we were exhaustifying below a universal operator or inside the antecedent of a conditional.)

While for CMs and SMs it is again relevant to consider the interaction with their implicatures from subdomain alternatives, the case of BNs is different as they do not have subdomain alternatives. I suggest the missing indirect scalar implicature of BNs could be due to a simple economy constraint on exhaustification blocking the exhaustification of *not BN* if it would lead to the same meaning as the exhaustification of *BN*:

(71)   *$O_{ScalAlts}$ $(\neg$ BN) if equivalent to $O_{ScalAlts}$ (BN)

To sum up, this analysis builds on the scalar implicature account for bare numerals and reclaims it for modified numerals also. The absence of scalar implicatures of CMs and SMs is explained through clashes with implicatures arising from their subdomain alternatives, in a way that will become clearer in the next section. The absence of the indirect implicature of BNs under negation is captured via competition with the direct implicature of an unembedded BN.

Bonus result: The existential implicature of *less than n P Q* and *at most n P Q* Alrenga (2016) (and references therein) notes that an utterance such as *LeBron scored at most 20 points in last night's game* typically carries an inference that he did score some points, that is, an existential inference, and also that such an inference can be cancelled with a continuation of the form *and it's even possible that he didn't score any points at all*, suggesting that it is an implicature. On our account this falls out straightforwardly from the idea that SMs, just like BNs or CMs, have scalar alternatives. While an implicature based on the scalar alternative *LeBron scored at most 19 points* leading to an 'exactly 20' meaning (at most 20 and not at most 19 = exactly 20) is blocked, implicatures based on any lower numeral alternative can

be safely excluded; this includes the *at most 0* alternative, leading to an implicature that he scored a non-zero number of points.

(72) $O_{ScalAlts}$ *(LeBron scored at most 20 points)*
        $O_{ScalAlts}$ $(|P \cap Q| \in [\![much]\!] (20))$
        $= |P \cap Q| \in [\![much]\!] (20) \wedge \neg |P \cap Q| \in [\![much]\!] (18/ \ldots /0)$
        'The number of points that LeBron scored was at most 20 but not at most 18 / ... / at most 0.'
        $=$ 'LeBron scored between 19 / ... / 1 and 20 points.'
        $\rightsquigarrow$ 'LeBron scored more than 0 points.'

We can make the same case for *less than n*.

Reclaiming scalar implicatures for modified numerals thus straightforwardly captures the existential inference of modified numerals with an upper-bounding entailment as a lower-bounding implicature.

2.2.3 Ignorance

Recall the empirical patterns: BNs never trigger ignorance inferences, in an unembedded context CMs do so optionally and SMs – obligatorily, when embedded under a universal modal such as *require* both CMs and SMs give rise to ignorance inferences optionally, and when embedded under negation CMs give rise to weak ignorance inferences. As already mentioned, SMs are generally bad under negation; we will address this issue separately in §2.2.4.

Similarly to earlier alternative-based accounts of the ignorance inferences of SMs such as Büring (2008), Mendia (2015), Kennedy (2015), or Spector (2015),[12] I show that the ignorance inferences of SMs can be derived as implicatures from exhaustification relative to subdomain alternatives. Going beyond these accounts, I also explain the BN and CM patterns.

More concretely, recall that BNs do not have subdomain alternatives. Thus, they cannot have exhaustification parses that would exploit such alternatives, and therefore they do not give rise to ignorance implicatures. CMs and SMs however do have subdomain alternatives, so they *can* have exhaustification parses that would exploit them. In fact, as proposed earlier in 2.1.3, the subdomain alternatives of CMs and SMs are exloited via $O^{PS}_{SubDomAlts}$ – that is, their exhaustification must lead to a properly stronger meaning (and must take into account not just the assertion but the presupposition-enriched assertion, but this will not be our main focus here; see §2.2.4). Moreover, the subdomain alternatives of SMs must always be exploited, that is, SMs cannot have any parse where they are not in the scope of $O^{PS}_{SubDomAlts}$. Now, whether a CM- or SM-utterance can give rise to ignorance implicatures or not depends on whether a certain $O^{PS}_{SubDomAlts}$ parse succeeds or not, and whether these ignorance implicatures are optional or obligatory depends on whether a given CM- or SM- utterance can have any parses other than that particular ignorance-inducing parse.

Before we consider any particular CM or SM parse, though, let us first make an observation that will allow us to discuss their exhaustification parses more compactly. The observation is that CMs and SMs both have truth conditions of the same form, namely $|P \cap Q| \in D$ (CMs: $|P \cap Q| \in \overline{[\![much/little]\!]}(n)$; SMs: $|P \cap Q| \in [\![much/little]\!](n)$), and also subdomain alternatives of the same form $|P \cap Q| \in D'$, $D' \subseteq D$ (CMs: $|P \cap Q| \in A$, $A \subseteq \overline{[\![much/little]\!]}(n)$; SMs: $|P \cap Q| \in A$, $A \subseteq [\![much/little]\!](n)$). Due to this formal similarity, we can discuss their $O^{PS}_{SubDomAlts}$ parses as one case.

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[12]For a full comparison with some of those accounts see §3.11 and §3.12. In general the difference between the existing pragmatic accounts of the ignorance inferences of SMs and the present account is the fact that the alternatives here are derived rather than stipulated, and the general reasoning is extended to CMs also.

Ignorance in unembedded environments   First, let's consider their parses in an unembedded environment. One parse would be a parse without $O^{PS}_{SubDomAlts}$, (73-a). This parse is available to CMs but not SMs, which must always be in the scope of $O^{PS}_{SubDomAlts}$. A second parse is one with $O^{PS}_{SubDomAlts}$, (73-b). This parse however fails because of contradiction: the assertion would be saying that $|P \cap Q|$ is a number in range D but the implicatures would be saying that it's not in any subset of D – clearly, an impossibility. A third parse is one where we try to rescue the second parse by inserting $\boxdot$ (our last resort, silent, matrix-level, universal epistemic modal) in between the exhaustivity operator and its prejacent, (73-c). This parse succeeds and leads to ignorance implicatures, since the result is a meaning that says that in all the speaker's epistemic states $|P \cap Q|$ is in D but not in all is it in A and not in all is it in B, etc.

(73)   Alice has more/less than three / at most/least three diamonds.

    a. $|P \cap Q| \in D$ $\boxed{\text{no } O^{PS}_{SubDomAlts} \text{ !}}$

    b. $O^{PS}_{SubDomAlts} (|P \cap Q| \in D)$
        $= |P \cap Q| \in D \wedge \neg(|P \cap Q| \in A) \wedge \neg(|P \cap Q| \in B)\ldots$, for all $A, B, \cdots \subset D$
        $= \bot$ $\boxed{\text{contradiction!}}$

    c. $O^{PS}_{SubDomAlts} \boxdot (|P \cap Q| \in D)$
        $= \boxdot |P \cap Q| \in D \wedge \neg \boxdot (|P \cap Q| \in A) \wedge \neg \boxdot (|P \cap Q| \in B)\ldots$, for all $A, B, \cdots \subset D$
        'The speaker is sure that the number of diamonds that Alice has is in D, but for any subset D' of D the speaker is not sure that the number is in that particular subset.'

                                                                              $\boxed{\text{ignorance implicatures}}$

In an unembedded environment CMs thus have two successful parses relative to their subdomain alternatives: one where their subdomain alternatives are not used, (73-a), and one where they are used and lead to ignorance, (73-c); this explains why ignorance implicatures are optional for CMs. In contrast, SMs only have only one successful parse relative to their subdomain alternatives, namely, the parse where these are used via $O^{PS}_{SubDomAlts}$, and where this leads to ignorance, (73-c); this explains the obligatoriness of their ignorance implicatures in this environment.

    Interaction with scalar implicatures   As suggested at the end of §2.2.2, in this unembedded context the ignorance implicatures of CMs and SMs interact in an interesting way with their scalar implicatures. (74) below shows an exhaustification parse of a CM-/SM-utterance where both their subdomain and their scalar alternatives are exploited simultaneously in the configuration that gave rise to ignorance implicatures in (73-c) and to the unattested scalar implicature in (55)-(56). Note that the prejacent to $O^{PS}_{SubDomAlts}$ is now $\boxdot O_{ScalAlts} (|P \cap Q| \in \{3, 4, \ldots\})$. $O^{PS}_{SubDomAlts}$ will assert this prejacent and negate all the stronger subdomain alternatives.[13]   The last row of the calculation gives the scalar-implicature-strengthened prejacent in the first term and the ignorance implicatures arising from $O^{PS}_{SubDomAlts}$ in the remaining terms. As we can see, the result is a contradiction, as the scalar implicature leading to the '$\boxdot$ exactly 3' meaning is in contradiction with all the implicatures from subdomain alternatives where D' is a set containing 3.

    (74)   Alice has more than two / at least three diamonds.
            $O_{SubDomAlts} \boxdot O_{ScalAlts} (|P \cap Q| \in \{3, 4, \ldots\})$

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[13]Note that the subdomain alternatives are things of the form $\boxdot(|P \cap Q| \in D')$, $D' \subset \{3, 4, \ldots\}$, without $O_{ScalAlts}$. That is because subdomain alternatives are defined relative to $|P \cap Q| \in D$, not relative to $O_{ScalAlts} (|P \cap Q| \in D)$.

$$= \boxdot O_{ScalAlts}\ (\underline{|P \cap Q| \in \{3, 4, \dots\}}) \ \wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{3\})} \ \wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{4, 7\})} \ \wedge$$
$$\underline{\neg \dots}$$
$$= \boxdot(\underline{|P \cap Q| \in \{3, 4, \dots\}} \ \wedge \ \neg |P \cap Q| \in \{4, 5, \dots\} \ \wedge \ \neg |P \cap Q| \in \{5, 6, \dots\} \ \wedge \neg \dots)$$
$$\wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{3\})} \ \wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{4, 7\})} \ \wedge \ \underline{\neg \dots}$$
$$= \boxdot(\underline{|P \cap Q| \in \{3\}}) \ \wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{3\})} \ \wedge \ \underline{\neg \boxdot (|P \cap Q| \in \{4, 7\})} \ \wedge \ \underline{\neg \dots}$$
$$= \bot$$

There are two logical ways to solve the contradiction: either to prune all the subdomain alternatives where D' is a set containing 3 and to keep the scalar implicature leading to the 'exactly 3' meaning, or to prune the scalar alternative leading to the 'exactly 3' meaning.

The first option is not viable: while pruning the offending subdomain alternatives would solve the contradiction, the result would still violate the proper strengthening presupposition of $O^{PS}_{SubDomAlts}$ because none of the remaining implicatures from subdomain alternatives would lead to a properly stronger meaning anyway.

The second option however works out: if we prune the scalar alternative leading to the 'exactly 3' meaning, then not only do we avoid the contradiction but there will always be some implicatures from subdomain alternatives that will lead to strengthening. Very concretely, if we remove $|P \cap Q| \in \{4, 5, \dots\}$, $O_{ScalAlts}$ yields $\boxdot(|P \cap Q| \in \{3, 4\})$, and the ignorance implicatures $\neg \boxdot (|P \cap Q| \in \{3\}$ and $\neg \boxdot (|P \cap Q| \in \{4\}$ would help satisfy the proper strengthening requirement of $O^{PS}_{SubDomAlts}$. Note also that the same discussion can be used to derive why CMs and SMs do not give rise to scalar implicatures in cases with a universal modal ($O_{SubDomAlts} \Box O_{ScalAlts}\ (|P \cap Q| \in D)$) or $O_{SubDomAlts}\ \boxdot O_{ScalAlts}\ (|P \cap \Box Q| \in D)$; see a discussion of the $O_{SubDomAlts}$ LFs below in the paragraph on ignorance under universal modals).

Note that for CMs this contradiction between the meaning producing via scalar implicatures and the meaning produced via ignorance implicatures can be avoided in the first place by simply not having $O^{PS}_{SubDomAlts}$ in the parse, namely, by interpreting CMs in a contexts where ignorance has been suspended; the prediction is that in that case we might see a CM giving rise to an 'exactly' meaning.

Note additionally that contexts with scale granularity greater than 1 implicitly prune the scalar alternative that would lead to an 'exactly n' meaning, which is why we find both CMs and SMs able to give rise to scalar implicatures in these cases.

Ignorance under a universal modal   Finally, let's consider the $O^{PS}_{SubDomAlts}$ parses available to a CM or SM embedded under a universal modal such as *require* (represented below as $\Box$). The first one is again a parse without exhaustification (thus, no implicatures), (75-a), which as before is available to CMs but not SMs. Then, a parse where $O^{PS}_{SubDomAlts}$ is below $\Box$, (75-b). Just like (73-b), this parse crashes because $O^{PS}_{SubDomAlts}$ would lead to contradiction. Unlike (73-b), it cannot be rescued via insertion of $\boxdot$ because $\boxdot$ can only be inserted at matrix level. Third, a parse where $O^{PS}_{SubDomAlts}$ is above $\Box$, (75-c), which does lead to implicature, but not of the ignorance type. Fourth, we have a parse where $O^{PS}_{SubDomAlts}$ is at the top again, but the numeral takes scope above *require*, (75-d). This parse crashes due to exhaustification leading to contradiction, for reasons entirely parallel to (73-b) or (75-b). However, like (73-b), it can be rescued via insertion of $\boxdot$, (75-e), leading to ignorance implicatures.

(75)   Alice is required to have more/less than three / at most/least three diamonds.

a. $\square(|P \cap Q| \in D)$ $\boxed{\text{no } O^{PS}_{SubDomAlts} \, !}$

b. $\square \, O^{PS}_{SubDomAlts} \, (|P \cap Q| \in D)$ $\boxed{\text{contradiction!}}$

c. $O^{PS}_{SubDomAlts} \, (\square(|P \cap Q| \in D))$
'In all the worlds compatible with what is required the number of diamonds that Alice has is in D, but not in all such worlds is it in subset A of D and not in all such worlds is it in subset B of D and …' $\boxed{\text{non-ignorance implicatures}}$

d. $O^{PS}_{SubDomAlts} \, (|P \cap \square Q| \in D)$ $\boxed{\text{contradiction!}}$

e. $O^{PS}_{SubDomAlts} \, (\boxdot(|P \cap \square Q| \in D))$
'The speaker is sure that the number of things that are diamonds and such that Alice is required to have them is in D, but for any subset D' of D the speaker is not sure that the number of such things is in that particular subset.' $\boxed{\text{ignorance implicatures}}$

Thus, when embedded under a universal modal both CMs and SMs have two successful $O^{PS}_{SubDomAlts}$ parses, only one of which, however, leads to ignorance implicatures. This explains why in this environment ignorance implicatures are optional not just for CMs but also for SMs.

Interaction with scalar implicatures In contexts where the numeral expression occurs under a universal modal the interaction of ignorance and non-ignorance implicatures from sub-domain alternatives with the scalar implicatures proceeds pretty much the same as discussed for ignorance in the unembedded case. In particular, both $O_{SubDomAlts} \square O_{ScalAlts} \, (|P \cap Q| \in D)$ and $O_{SubDomAlts} \boxdot O_{ScalAlts} \, (|P \cap \square Q| \in D)$ would lead to a clash resolved in favor of removing the offending scalar implicature / pruning the offending scalar alternative.

CMs can again also have a parse without $O^{PS}_{SubDomAlts}$. This means that *Alice is required to have more than three diamonds* can in fact have a parse such as $\square O_{ScalAlts} \, (|P \cap Q| \in D)$ – *Alice is required to have (more than three and not more than four = exactly four) diamonds* (). I surmise that in this case the truth conditions simply leave it open that that could be the case.

Ignorance under negation Let us now consider ignorance implicatures in the case where the modified numeral is embedded under negation. As before, we will leave SMs aside for §2.2.4 and focus just on CMs. The following LFs are conceivable. First, an LF without $O^{PS}_{SubDomAlts}$, (76), which is okay for CMs. Then, an LF with $O^{PS}_{SubDomAlts}$, which crashes due to the fact that all the subdomain alternatives $\boxdot\neg(|P \cap Q| \in D')$, $D' \subseteq D$ are entailed, so exhasutification is vacuous and thus fails to satisfy the proper strengthening requirement of $O^{PS}_{SubDomAlts}$. Third, an LF where we try to rescue the previous LF by insertion of $\boxdot$; this, however, fails also for the exact same reason as the one before, and, unlike in the positive case, no ignorance implicatures are derived.

(76)   Alice doesn't have more/less than three diamonds.
a. $\neg(|P \cap Q| \in D)$
b. $O^{PS}_{SubDomAlts} \, \neg(|P \cap Q| \in D)$
$= \boxdot\neg(|P \cap Q| \in D)$ $\boxed{\text{no proper strengthening!}}$

c. $O^{PS}_{SubDomAlts} \, \boxdot \, \neg(|P \cap Q| \in D)$
$= \boxdot\neg(|P \cap Q| \in D)$ $\boxed{\text{no proper strengthening!}}$

We thus predict CMs to give rise to no ignorance implicatures under negation. Yet, as we noted in §1.3, it feels like (76) does give rise to some weak ignorance inferences of the form *The speaker is not sure whether she has three or two or . . . / three or four or . . . diamonds*. I surmise that these weak ignorance inferences are not in fact formally sanctioned but appear merely due to the fact that the truth conditions leave matters unsettled. This would also explain their weak character.

To sum up, we have captured the ignorance inferences of CMs and SMs as implicatures arising from the interaction of $O^{PS}_{SubDomAlts}$ and $\boxdot$. Their obligatoriness in unembedded contexts for SMs comes from the fact that they cannot have parses where their subdomain alternatives are not used, that is, they must always be in the scope of $O^{PS}_{SubDomAlts}$; their optionality for both CMs and SMs under a universal modal – from the fact that in that environment there are two successful $O^{PS}_{SubDomAlts}$ parses of which only one leads to ignorance; and their absence when the CM (or SM) is embedded under negation – from the fact that the ignorance implicature-inducing parse crashes due to failure to meet proper strengthening.

Bonus result: The missing reading of SMs under an existential modal   The literature on modified numerals (e.g., Geurts and Nouwen 2007, Nouwen 2010, Coppock and Brochhagen 2013, Kennedy 2015, a.o.) note an asymmetry between CMs and SMs with respect to their possible readings under an existential modal. To use an example from Coppock and Brochhagen (2013:4), the claim is that (77-a) below can be used to grant permission to have fewer than three beers but (77-b) can only be used to forbid having more than two.

(77)   a.   You may have less than three beers.
        b.   You may have at most two beers.

I will argue that this asymmetry again falls out as a consequence of the allowed CM- and SM-parses in this environment. Consider again the various possible parses. First, there is a parse where the modal takes scope above the numeral and there is no $O^{PS}_{SubDomAlts}$, (78-a). As we already know, such a parse is available to CMs but not SMs. Second, a parse where we exhaustify below the modal, (78-b), which however fails due to contradiction. Third, one where we exhaustify above the modal, (78-c), which similarly fails due to contradiction. (We would be saying that there is a world where the number is in D but there is no world where it is in subset A of D and no world where it is in subset B of D etc. – a contradiction.) Note that we can't rescue this parse via $\boxdot$ because the parse already contains a matrix level modal – the existential modal. Fourth, a parse where the CM or SM takes scope above the modal and there is no $O^{PS}_{SubDomAlts}$, (78-d). As always, SMs can't have such a parse. Fifth, a parse where we have the same scope reading but also $O^{PS}_{SubDomAlts}$, (78-e). Such a parse fails due to contradiction. However, we can rescue it by inserting $\boxdot$, (78-f), which produces ignorance implicatures about the exact allowable number.

(78)   You may have less than three / at most two beeers.

   a. $\Diamond\,(|P \cap Q| \in \{0, 1, 2\})$ $\boxed{\text{no } O^{PS}_{SubDomAlts}\ !}$

   b. $\Diamond\,O^{PS}_{SubDomAlts}\,(|P \cap Q| \in \{0, 1, 2\})$ $\boxed{\text{contradiction!}}$

   c. $O^{PS}_{SubDomAlts}\,\Diamond\,(|P \cap Q| \in \{0, 1, 2\})$ $\boxed{\text{contradiction!}}$

   d. $(|P \cap \Diamond\,Q| \in \{0, 1, 2\})$ $\boxed{\text{no } O^{PS}_{SubDomAlts}\ !}$

   e. $O^{PS}_{SubDomAlts}\,(|P \cap \Diamond\,Q| \in \{0, 1, 2\})$ $\boxed{\text{contradiction!}}$

f. $O^{PS}_{SubDomAlts} \boxdot (|P \cap \Diamond Q| \in \{0, 1, 2\})$

'The speaker is sure that the number of things that are beers and such that you may have them is in $\{0, 1, 2\}$, but for any subset of $\{0, 1, 2\}$ the speaker is not sure that the number is in that particular subset.' $\boxed{\text{implicatures}}$

As usual, let's recap which of these parses are possible for CMs as opposed to SMs. CMs can have the non-exhaustification parse (78-a) granting permission to have 0, 1, or 2 beers, parse (78-d) saying that the number of beers such that you can have that many beers is in $\{0, 1, 2\}$ and thus fordbidding more than two, and (78-f) entailing the same but additionally triggering ignorance implicatures. Of these parses SMs can only have the last one, which is why they do not convey permission to have less than three but do convey interdiction to have more than two. This solves the missing readings puzzle.

To sum up, our assumptions about the different status of the subdomain alternatives of CMs and SMs not only helped us capture exactly the right ignorance patterns in unembedded and emebdded contexts but also helped us make sense of why when embedded under existential modals SMs seem to miss a reading. Aside from that, considering preexhaustified alternatives helped us derive epistemic implicatures of flavors other than ignorance also.

2.2.4 Acceptability in downward-entailing environments

Recall that in the scope of negation BNs and CMs are okay but SMs are degraded. All are fine in the antecedent of a conditional or the restriction of a universal operator.

I propose that, just like ignorance, this pattern of (in)felicity in downward-entailing environments arises from exhaustification relative to subdomain alternatives.[14] BNs do not have this type of alternatives so they are not affected, which is why they are felicitous. As for CMs and SMs, in a plain downward-entailing environment such as the scope of negation all their $O^{PS}_{SubDomAlts}$ parses fail (because of failure to lead to proper strengthening, as already anticipated in (76)), but in a presuppositional downward-entailing environment such as the antecedent of a conditional or the restriction of a universal there is one $O^{PS}_{SubDomAlts}$ parse that succeeds. SMs don't have the option of a non-exhaustification parse, so the fact that there is no succeessful $O^{PS}_{SubDomAlts}$ parse in the scope of negation but there is one such parse in an antecedent or a restrictor results in infelicity in the former downward-entailing environment environment but felicity in the latter.

Let's consider first the case of embedding under negation. First, there is again the non-$O^{PS}_{SubDomAlts}$ parse, (79-a), available to CMs but not SMs. Second, the parse where $O^{PS}_{SubDomAlts}$ is below negation, (79-b). This fails due to contradiction (for the same reason as discussed in the case of ignorance). Third, the parse where $O^{PS}_{SubDomAlts}$ is above negation, (79-c). This fails because all the subdomain alternatives are entailed by the assertion, so exhaustification is vacuous, which means no proper strengthening. Fourth, inserting $\boxdot$ in between the exhaustivity operator and negation, (79-d), does not help – the subdomain alternatives are still entailed, so exhaustification still fails to lead to proper strengthening.

(79) Alice doesn't have more/less than three / at most/least three diamonds.

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[14]This follows an existing solution for the unacceptability of SMs in the scope of negation/downward-entailing environments by Spector (2015). I also extend the solution to presuppositional downward-entailing environments based on previous suggestions by Chierchia (2013), Spector (2014), and Nicolae (2017) (all of whom look at the interaction between exhaustification and presuppositional environments as a way to capture the behavior of certain polarity sensitive items such as strong NPIs and strong PPIs).

a. $\neg(|P \cap Q| \in D\})$ no $O^{PS}_{SubDomAlts}$!

b. $\neg O^{PS}_{SubDomAlts} (|P \cap Q| \in D)$ contradiction!

c. $O^{PS}_{SubDomAlts} \neg(|P \cap Q| \in D)$ no proper strengthening!

d. $O^{PS}_{SubDomAlts} \boxdot \neg(|P \cap Q| \in D)$ no proper strengthening!

SMs are thus bad in the scope of negation because all $O^{PS}_{SubDomAlts}$ parses fail. CMs can have the parse without $O^{PS}_{SubDomAlts}$, which is why they are acceptable in this environment.

Let us consider now the case of a downward-entailing environment such as the antecedent of a conditional or the restriction of a universal. Such environments are similar to the scope of negation in what regards monotonicity but differ from it in the sense that they additionally contain an existential presupposition in their first argument. For example, an utterance such as (26) below not only asserts that every x who has a certain number of diamonds wins, (80-a), but also presupposes that there are such x, (80-b).

(80) Everyone who has more/less than three / at most/least three diamonds wins.

    a. Assertion: $\forall x[\text{people}(x) \wedge |[\![\text{diamonds}]\!] \wedge [\![\text{objects-}x\text{-has}]\!]| \in D] \rightarrow \text{win}(x)]$
    b. Presupposition: $\exists x[\text{people}(x) \wedge |[\![\text{diamonds}]\!] \wedge [\![\text{objects-}x\text{-has}]\!]| \in D]$

Now, remember that the subdomain alternatives of both CMs and SMs are exhaustified via *strong* exhaustification, that is, exhaustification that takes into account the *presupposition-enriched* content of the assertion. This detail did not matter so far because we haven't been dealing with presuppositional prejacents. However, it matters now, because it means that the $O^{PS}_{SubDomAlts}$ parse of (80) where we exhaustify at the very top, that is, $O^{PS}_{SubDomAlts}$ (*Everyone...*), must proceed not with respect to the plain assertion, (80-a), but rather with respect to the presupposition-enriched assertion, (81-a), and with respect to subdomain alternatives based on it, (81-b).

(81)   a. Presupposition-enriched assertion, schematic form:
       $\forall x[\text{the \# of diamonds } x \text{ has } \in D \rightarrow \dots] \wedge \exists x[\text{the \# of diamonds } x \text{ has } \in D]$
    b. Subdomain alternatives to the presupposition-enriched assertion, schematic form:
       $\forall x[\text{the \# of diamonds } x \text{ has } \in D' \rightarrow \dots] \wedge \exists x[\text{the \# of diamonds } x \text{ has } \in D']$, where $D' \subset D$

Let's compare the presupposition-enriched assertion and its subdomain alternatives more closely. Note that while the $\forall$ part of the presupposition-enriched assertion entails the $\forall$ part of the subdomain alternative, the $\exists$ part of the presupposition-enriched assertion does not entail the $\exists$ part of the subdomain alternative but is rather entailed by it:

(82)
$$\begin{array}{ccc}
\forall x[\text{the \# of diamonds } x \text{ has } \in D \rightarrow \dots] & \wedge & \exists x[\text{the \# of diamonds } x \text{ has } \in D] \\
\Downarrow & & \Uparrow \\
\forall x[\text{the \# of diamonds } x \text{ has } \in D' \rightarrow \dots] & \wedge & \exists x[\text{the \# of diamonds } x \text{ has } \in D']
\end{array}$$

Note that none of the alternatives to the presupposition are thus entailed by it, so they must be false. However, negating them leads to contradiction because we'd be negating the conjunction of the $\forall$ part and of the $\exists$ part, (83-a), which means that either the $\forall$ part is false or the $\exists$ part is false, (83-b). The $\forall$ part however cannot be false because that would contradict the prejacent, so we are left with a series

of statements to the effect that it is not the case that there exists an x such that the number they have is in subset A of D and there doesn't exist an x such that the number they have is in subset B of D and so on, (83-c), which together would contradict the presupposition of the prejacent which says that that there is an x such that the number they have is somewhere in D.

(83)   a. $\neg(\forall x[\text{the \# of diamonds } x \text{ has } \in D' \to \dots] \land \exists x[\text{the \# of diamonds } x \text{ has } \in D'])$
       b. $\neg(\forall x[\text{the \# of diamonds } x \text{ has } \in D' \to \dots]) \lor \neg(\exists x[\text{the \# of diamonds } x \text{ has } \in D'])$
       c. $\neg(\exists x[\text{the \# of diamonds } x \text{ has } \in D']$

So $O^{PS}_{SubDomAlts}$ (*Everyone* ...) fails. However, we can, as usual, try to rescue it via insertion of $\boxdot$ – $O^{PS}_{SubDomAlts}$ $\boxdot$ (*Everyone* ...), (84-a)-(84-c). This time our subdomain alternatives can be excluded, leading to a strengthened meaning of the form 'The speaker is sure that everyone who owns $n \in D$ diamonds wins and that there exists someone who has $n \in D$ diamonds but not sure that there exists someone whose number of diamonds is in $A \subset D$, and not sure that there exists someone whose number of diamonds is in $B \subset D$, and so on for all the subsets of D.

(84)   a. $\neg \boxdot (\forall x[\text{the \# of diamonds } x \text{ has } \in D' \to \dots] \land \exists x[\text{the \# of diamonds } x \text{ has } \in D'])$
       b. $\neg\boxdot(\forall x[\text{the \# of diamonds } x \text{ has } \in D' \to \dots]) \lor \neg\boxdot(\exists x[\text{the \# of diamonds } x \text{ has } \in D'])$
       c. $\neg \boxdot (\exists x[\text{the \# of diamonds } x \text{ has } \in D']$

Thus, when an SM is in a *presuppositional* downward-entailing environment there *is* a successful $O^{PS}_{SubDomAlts}$ parse. This captures why SMs are acceptable in this type of environments.

   To sum up, SMs are bad in the scope of negation because they have no felicitous $O^{PS}_{SubDomAlts}$ parse in that configuration. They are however good in the antecedent of a conditional or the restriction of a universal because of the effect of the presupposition of these environments on $O^{PS}_{SubDomAlts}$.

2.2.5   Summary

Our theoretical setup in §2.1 derives all the basic patterns of BNs, CMs, and SMs with respect to entailments, scalar implicatures, ignorance implicatures, and acceptability in downward-entailing environments.

2.3   Additional results

In addition to capturing the empirical patterns discussed so far, any theory of BNs, CMs, and SMs should also

   ⋆ be compositional in a way that does justice to their morphological makeup

   ⋆ capture their predicative uses

   ⋆ yield a plausible constituent structure

Below I discuss how our theory fares with respect to all of these new desiderata.

2.3.1   Compositionality

Our lexical entries for n, *much/little*, [comp], and [sup] as given in §2.1.1 help us satisfy the compositionality desideratum at a number of levels.

First, as shown in detail in the previous sections, they not only help us to compositionally put together the meanings of BNs, CMs, and SMs, §2.1.1, but also give us a natural and uniform way to derive their scalar and subdomain alternatives (and their optional vs. obligatory status), §2.1.2, and ultimately allow us to account for all their main empirical puzzles, §2.2.

Second, they link up naturally to their meanings elsewhere in language. On our theory the numeral is a degree or a predicate; the predicative meaning goes back to Russell, and the degree meaning is often used in theories of numerals, especially as part of the meaning of modified numerals (e.g., Kennedy (2015) or Buccola and Spector (2016)). Then, *much* and *little* are extent indicators; this parallels the extent semantics of gradable adjectives as given by Seuren (1984) or Kennedy (1997). Third, our meaning for [comp], (85-a), can be restated into a Seuren (1984)-style A-not-A meaning, (85-b), or into into a Heim (2007)-style ⊂ relation, (85-c). Finally, our meaning for [sup], (86-a), can be restated into a Hackl (2009)-style sort of meaning, (86-b) (or, for that matter, it can also be expressed as a relation, (86-c), just like [comp]). (Our original statement of the meanings of [comp] and [sup] is however more suitable for our purposes since it allows us to see more clearly the connection between the truth conditions of CMs and SMs and their subdomain alternatives.) Of course, it remains to be seen to what extent the manifestations of these morphemes in BNs, CMs, and SMs are similar to their manifestations elsewhere in language.

(85)   〚more/less than n P Q〛

    a. $|P \cap Q| \in \overline{〚\text{much/little}〛(n)}$

    b. $\exists d[〚\text{much/little}〛(|P \cap Q|)(d) \wedge \neg 〚\text{much/little}〛(n)(d)]$
        'There is a degree d such that it in the positive/negative extent of $|P \cap Q|$ but not in the positive/negative extent of n.'

    c. $〚\text{much/little}〛(3) \subset 〚\text{much/little}〛(|P \cap Q|)$

(86)   〚at most/least n P Q〛

    a. $|P \cap Q| \in 〚\text{much/little}〛(n)$

    b. $\forall d[d \in 〚\text{much/little}〛(|P \cap Q|) \wedge d \neq n \rightarrow n > / < d]$
        'For all degrees d, if d is in the positive/negative extent of $|P \cap Q|$ and it is different from n, then n is strictly greater/smaller than d.'

    c. $〚\text{much/little}〛(n) \supseteq 〚\text{much/little}〛(|P \cap Q|)$

Finally, our lexical entries for the morphological pieces of BNs, CMs, and SMs also ensure that the BN-, CM-, or SM-modified DP will pose no challenges to further compositionality either. That is because *three P* (after existential closure), *more/less than three P*, *at most/least three P* are all generalized quantifiers. This means that moving them (e.g., as we did for scope interactions), or combining them with other parts of an utterance (e.g., conjoining/disjoining them to another DP: *every chicken and at least three ducks*) can proceed in a completely standard way. (See §3.1 and §3.3 for a discussion of how our approach compares to GQT and how it handles Hackl 2000's criticism of GQT.)

2.3.2   Predicative uses of bare and modified numerals

In our theory *n P* (after existential closure), *more/less than n P*, *at most/least n P* are all generalized quantifiers. As pointed out in §2.3.1, this is an advantage because it allows us to reap the benefits of such an analysis (no surprises with movement, straightforward ability to conjoin/disjoin them to any other DP). At the same time, it also raises the question of how to put together compositionally the

sentences in (87)-(89), or how to get the long distance reading of (90). Note that all of these questions could be answered easily if *n P*, *more/less than n P*, and *at most/least n P* were mere predicates.

(87) The three / more/less than three / at most/least three NP

(88) We are three / more/less than three / at most/least three.

(89) Plant a tree every three houses.

(90) If two relatives of mine die, I'll be rich.

The answer for BNs is easy – before we do existential closure, *n P is* in fact just a mere predicate.

The same is not true of *more/less than n P* and *at most n P*, which are generalized quantifiers. I propose that in cases such as the above these items undergo typeshifting via Partee (1987)'s BE, (91), which turns them from generalized quantifiers into predicates. Below I show this for *at most three students*, (92).
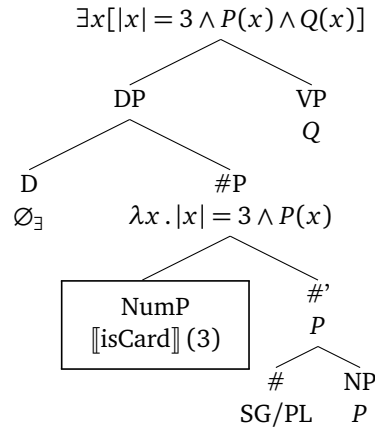
(91) $[\![\text{BE}]\!] = \lambda Q_{\langle \alpha t, t \rangle} . \lambda x_\alpha . Q(\lambda y_\alpha . y = x)$

(92) $[\![\text{BE}]\!] \, ([\![\text{at most three students}]\!])$
$= [\lambda Q_{\langle et, t \rangle} . \lambda x_e . Q(\lambda y_e . y = x)](\lambda Q_{\langle e, t \rangle} . |P \cap Q| \in [\![\text{much}]\!] \, (3))$
$= \lambda x_e . [\lambda Q_{\langle e, t \rangle} . |P \cap Q| \in [\![\text{much}]\!] \, (3)](\lambda y_e . y = x)$
$= \lambda x_e . |P \cap \lambda y_e . y = x| \in [\![\text{much}]\!] \, (3)$

Thus, *n P* is at origin a predicate but can become a generalized quantifier via existential closure, while *more/less than n P* and *at most/least n P* are at origin generalized quantifiers but can become predicates via BE. This captures all their uses.

2.3.3 Constituency

Our semantics for BNs, CMs, and SMs also suggests a certain constituent structure. Is it a plausible one?

First, let's discuss the case of BNs. According to our semantics, *n* (typeshifted into a predicate) first combines with *P*, then the *n P* complex undergoes existential closure. This could go with a syntactic structure as the one below. First, a number head # with a singular/plural meaning ($[\![\text{SG}]\!] = \lambda P_{\langle e, t \rangle} : \forall x \in P[|x| = 1].P$, $[\![\text{PL}]\!] = \lambda P_{\langle e, t \rangle} . P$), cf. Scontras 2013) takes as a complement the NP with the meaning *P*. Semantically this corresponds merely to a presupposition check on *P*, the result at node #' being still *P*. Second, the numeral phrase NumP with the meaning $[\![\text{isCard}]\!] \, (3)$ is merged in the specifier position of #P. Semantically this corresponds to $[\![\text{isCard}]\!] \, (3)$ and *P* combining via predicate modification (Heim and Kratzer 1998) to yield another predicative meaning. This meaning is then closed via a silent existential quantifier hosted in D.

$$\exists x[|x| = 3 \land P(x) \land Q(x)]$$

Tree structure:
- Root: $\exists x[|x| = 3 \land P(x) \land Q(x)]$
  - DP
    - D: $\varnothing_\exists$
    - #P: $\lambda x \,.\, |x| = 3 \land P(x)$
      - NumP: $[\![\text{isCard}]\!]\,(3)$
      - #': P
        - #: SG/PL
        - NP: P
  - VP: Q

As for modified numerals, our semantics has [comp]/[sup] taking as arguments first *much*/*little*, then the numeral n, then P, then Q. This could fit into a syntactic structure identical to the one above modulo the fact that now NumP is the complement of Mod, the result of putting together [comp]/[sup] and *much*/*little*. Semantically this corresponds to feeding [comp]/[sup] their n argument. The resulting ModP combines with #', and the resulting #P combine with the VP, which semantically corresponds to feeding [comp]/[sup] their P and Q arguments. The D is either null and projected merely for syntactic reasons, or not projected at all.

Tree structure:
- Root: $[\![\text{comp/sup}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$
  - #P: $\lambda Q \,.\, [\![\text{comp/sup}]\!]\,([\![\text{much/little}]\!])(n)(P)(Q)$
    - ModP
      - Mod
        - [comp]/[sup]
        - much/little
      - NumP: n
    - #': P
      - #: SG/PL
      - NP: P
  - VP: Q
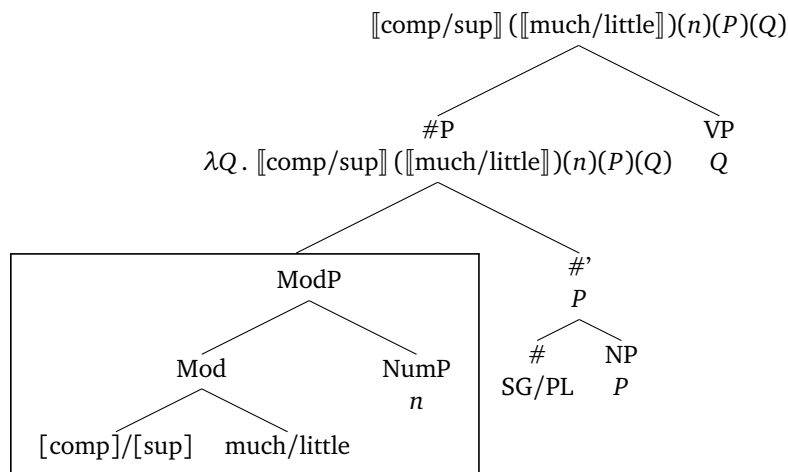
The structures above imply that n forms a constituent with P in BNs but with the modifier in CMs and SMs. Is this plausible?

Krifka (1999) notes that the comparative and superlative modifiers can associate with phrases of different sizes via focus, for example with numerals, DPs, APs, VPs, or Det's. Below I illustrate this for *at least* (examples adapted from Krifka 1999, with some additions).

(93)   a.   At least [three]$_F$ boys left.
        b.   At least [three boys / John and Mary / the first year students]$_F$ left.
        c.   Mary was at least [satisfied]$_F$.
        d.   The guest at least [left early]$_F$.
        e.   At least [some]$_F$ determiners aren't determiners.

This association is reflected in the sort of questions each of these examples can answer. (93-a) can answer a question of the form *How many boys left?*, with the modified numeral being able to act as a short answer: *At least three*. (93-b) answers a question of the form *Who left?* And so on.

This association is also reflected in the sort of modified units that we can conjoin. For example, the modified numeral in (93-a) can be conjoined with another modified numeral as in *[More than three but less than six] people attended*.

I conclude thus that in each of these cases the modifier forms a constituent with the focused phrase, acting as a degree modifier+quantifier for CMs and SMs, a DP-modifier, an AP modifier, a VP modifier, and a determiner modifier, respectively. In particular, that in CMs and SMs the modifier forms a constituent with the numeral, as in the tree above.

While this means that our story for comparative and superlative modifiers is plausible, it also means that it is incomplete, since to account for (93-b)-(93-e), where the modifier forms a constituent with things other than a numeral, we would have to modify the meanings of [comp]/[sup] and *much/little*. Intuitively, the effect of the modifier in those cases is analogous to its effect in modified numerals. For example, *At least John and Mary left* could be understood as saying that the largest plurality in the denotation of ⟦*left⟧ is a member of the set of individuals smaller than or equal to the sum individual formed by John and Mary (essentially extending extents to individuals). However, to obtain this meaning we would have to adjust the types of [comp]/[sup] and generalize our notion of extents. I leave this for future research.

2.3.4 Summary

To sum up, our theory also offers plausible solutions to important additional desiderata for a theory of BNs, CMs, and SMs such as compositionality, predicative uses, and constituency.

2.4 Conclusion

In this section I proposed a theory of BNs, CMs, and SMs that

* captures their entailments, scalar implicatures, ignorance implicatures, acceptability in downward-entailing environments

    + explains additional puzzles such as the existential implicature of modified numerals with an upper-bounding entailment or the missing readings of SMs under an existential modal

* is compositional in a way that takes into account the morphological makeup of BNs, CMs, and SMs), can handle the predicative uses of BNs, CMs, and SMs, and also offers a plausible constituent structure.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓	✓	✓	✓	✓	✓

Moreover, it achieves all of these from nothing more than the lexical meanings of BNs, CMs, and SMs and general, independently motivated semantic and pragmatic mechanisms.

In the next section we will compare this account with a number of other existing accounts. As we will see, none of them cover as much ground.

3   Comparison to previous literature

The existing literature on numerals is vast and diverse. The various theories published on this topic vary a lot in motivating data, goals, scope, framework, and other ways. It would be impossible to do each of them justice on their own terms. In what follows I will therefore confine myself to what I believe is a representative sample of the theories that have been proposed so far, and I will assess each of them using the same measuring stick that we have developed so far – that is, with respect to entailments, scalar implicatures, ignorance implicatures, acceptability in downward-entailing environments, compositionality, predicative uses, and constituency.

The discussion of each article is structured as follows. First I give a brief overview of the theory. Then I discuss what it gets and what it doesn't, finishing up with a little table. In the table '✓' indicates that the theory meets a certain desideratum, '✗' – that it does not, '?' – that it does not address it, '✓/✗' – that it captures some of the patterns but does not capture other patterns and is in fact set up such that it is unable to capture them, and '✓/?' that it captures certain patterns but does not address others, though it is not obvious that it would be unable to do so if suitably extended.

The various articles are related to each other in complex ways, and there is no one best way to organize them. For that reason I have chosen to discuss them first in isolation and in chronological order. However, at the end, in section §3.14, I offer a global summary where I organize the discussion by desiderata and group the theories by the strategies they adopt to address a certain desideratum.

3.1   Horn (1972) + Barwise and Cooper (1981)

Barwise and Cooper (1981)'s Generalized Quantifier Theory (GQT) treats BNs, CMs, and SMs as relations between a predicate with a nominal meaning P and a predicate with a verbal meaning Q, with truth conditions as below. Note that implicit in Barwise and Cooper's meaning for bare numerals is Horn (1972)'s view that BNs entail a lower bound and acquire their upper bound via scalar implicature.

(94)

⟦three ⟧	$= \lambda P . \lambda Q . \lvert P \cap Q \rvert \geq 3$
⟦three students⟧	$= \lambda Q . \lvert \llbracket \text{students} \rrbracket \cap Q \rvert \geq 3$
⟦more than three⟧	$= \lambda P . \lambda Q . \lvert P \cap Q \rvert > 3$
⟦more than three students⟧	$= \lambda Q . \lvert \llbracket \text{students} \rrbracket \cap Q \rvert > 3$
⟦less than three⟧	$= \lambda P . \lambda Q . \lvert P \cap Q \rvert < 3$
⟦less than three students⟧	$= \lambda Q . \lvert \llbracket \text{students} \rrbracket \cap Q \rvert < 3$
⟦at most three⟧	$= \lambda P . \lambda Q . \lvert P \cap Q \rvert \leq 3$
⟦at most three students⟧	$= \lambda Q . \lvert \llbracket \text{students} \rrbracket \cap Q \rvert \leq 3$
⟦at least three⟧	$= \lambda P . \lambda Q . \lvert P \cap Q \rvert \geq 3$
⟦at least three students⟧	$= \lambda Q . \lvert \llbracket \text{students} \rrbracket \cap Q \rvert \geq 3$

This theory gets the entailments. It also gets the scalar implicatures, although it doesn't address all the cases where scalar implicatures seemed to be missing. It doesn't address ignorance implicatures,

acceptability in downward-entailing environments, compositionality, or predicative uses. Constituency is the same as ours, minus the syntactic details.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓/?	✓/?	?	?	?	✓

Our theory shares a number of features with GQT. With the exception of BNs, we derive the bounding entailments in the same way, by direct reference to $|P \cap Q|$. We also maintain Horn's original notion of scalar alternatives. In addition to this, we preserve the GQT idea that *three P*, *more/less than three P*, and *at most/least three P* are generalized quantifiers, and that CMs and SMs are thus natural language determiners. Note, however, that, unlike GQT, our theory does not treat numeral expressions as simplex but complex, building them up from their morphological pieces and factoring in the contribution of *much/many* or [comp]/[sup] – thus answering one of the main challenges to GQT from Hackl (2000) (for a fuller comparison see §3.3) as well as accounting for all the patterns that GQT did not address. In sum, our theory can be regarded as a direct expansion and refinement of GQT that not only recognizes the similarity of BNs, CMs, and SMs to one another and to other natural language determiners, but also their main differences.

3.2 Krifka (1999)

Krifka (1999) treats *n P*, *more/less than n P*, and *at most/least n P* on a par as predicates. The main difference between them lies with the status of the scalar alternatives introduced by the numeral. For *n P* these are active. In contrast, for *more/less n P* and *at most n P* these are neutralized through the action of the modifier, which operates on ordered sets of alternatives (partial order – $\leq$ – for SMs and total order – $<$ – for CMs) is defined such that, when applied to an argument α, it performs the union of the alternatives β that stand in a particular ordering relation to α. These predicative meanings then go on to saturate the first argument of a silent existential quantifier.

(95)   ($[\![\alpha_A]\!]$ = the alternative value of α, defined in terms of a strength relation over alternatives)

$[\![\text{three students}]\!]$ $\qquad = \lambda x . [|x| = 3 \wedge \text{students}(x)]$

$[\![\text{more than } \alpha]\!]$ $\qquad = \bigcup \{\beta \mid \langle \alpha, \beta \rangle \in [\![\alpha_A]\!]\}$
$[\![\text{more than [three students]}]\!]$ $\quad = \lambda x . [|x| > 3 \wedge \text{students}(x)]$

$[\![\text{less than } \alpha]\!]$ $\qquad = \bigcup \{\beta \mid \langle \beta, \alpha \rangle \in [\![\alpha_A]\!]\}$
$[\![\text{less than [three students]}]\!]$ $\quad = \lambda x . [|x| < 3 \wedge \text{students}(x)]$

$[\![\text{at most } \alpha]\!]$ $\qquad = \bigcup \{\beta \mid \langle \beta, \alpha \rangle \in [\![\alpha_A]\!]\}$
$[\![\text{at most [three students]}]\!]$ $\quad = \lambda x . [|x| \leq 3 \wedge \text{students}(x)]$

$[\![\text{at least } \alpha]\!]$ $\qquad = \bigcup \{\beta \mid \langle \alpha, \beta \rangle \in [\![\alpha_A]\!]\}$
$[\![\text{at least [three students]}]\!]$ $\quad = \lambda x . [|x| \geq 3 \wedge \text{students}(x)]$

From this setup Krifka derives scalar implicatures for BNs but not for CMs or SMs; this is a result he wants. On the other hand, the existential quantifier leads to a wrong lower-bounding rather than

upper-bounding entailment for *at most n P Q* and *less than n P Q*. To fix that, Krifka modifies the story to say that such utterances do not have any meaning proper (no truth conditions) but merely indicate that certain alternatives, namely the alternatives above the bound, are excluded by the speaker (falsity conditions).

While it does get the entailments and one of the scalar implicature patterns, this theory is however unable to capture the other scalar implicature patterns. In addition to this, it doesn't address ignorance or acceptability in downward-entailing environments.[15]

Finally, the theory is partly compositional (the contribution of the numeral is the same in all of BNs, CMs, and SMs) but doesn't engage with *much/little*, [comp]/[sup]. It respects constituency in the sense that *at least n P* form a constituent, but the modifier and the numeral do not form a constituent, which, while not obviously wrong, raises the question as to how they are able to act as a unit in short answers or to be conjoined with other modified numerals.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓/✗	?	?	✓/?	✓	✓/?

An important point of difference between this theory and our own is the way we derive the upper-bounding entailments of *less than* and *at most*. To achieve that, Krifka resorts to a non-standard notion of falsity conditions. Note however that the conceptual status of falsity conditions is debatable (see, e.g., Coppock and Brochhagen 2013 for criticism). In contrast, following Generalized Quantifier Theory, we get the same results both more straightforwardly and with more standard assumptions.

3.3 Hackl (2000)

Hackl (2000) proposes that a bare numeral denotes a simple degree, but modified numerals denote generalized quantifiers over degrees. They are all generated as sisters to a silent *many* which at the same time acts as a link between the numeral and the NP *and* introduces existential quantification over individuals, (96). Modified numerals can't be interpreted in situ due to type mismatch so they must move, which gives rise to a degree predicate at the movement site, (97). The resulting meanings for bare and modified numerals are as in (98).

(96) $[\![many]\!] = \lambda n_d . \lambda P_{\langle e,t \rangle} . \lambda Q_{\langle e,t \rangle} . \exists x[|x| = n \wedge P(x) \wedge Q(x)]$

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[15]Although in Cohen and Krifka 2014 the idea that *at most n P Q* gets its truth conditions via implicature is used to explain why they are bad in the scope of negation.

(97)
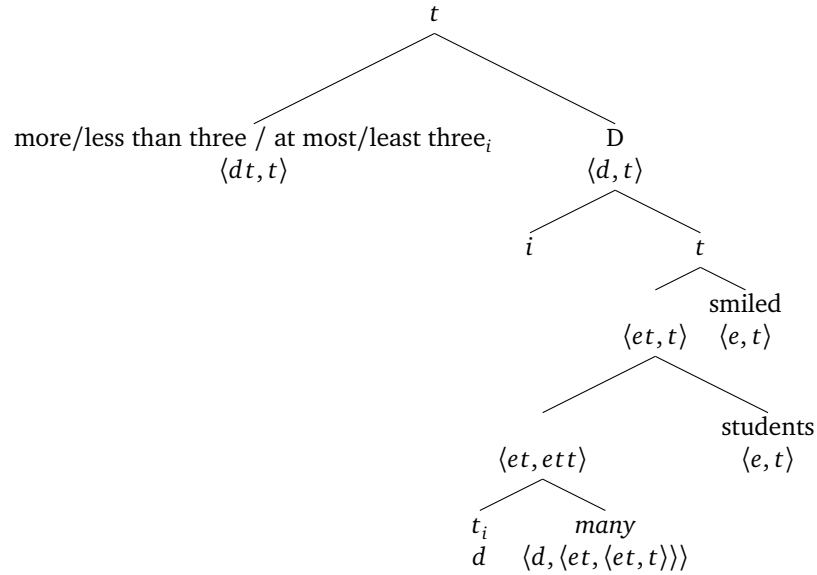
(98)   (below, max is defined as $= \lambda D_{\langle d,t \rangle} . \iota d$ s. t. $D(d) = 1 \wedge \forall d'[D(d') = 1 \to d' \leq d])$

$[\![$three$]\!]$	$= 3$
$[\![$three students smiled$]\!]$	$= \exists x[|x| = 3 \wedge \text{students}(x) \wedge \text{smiled}(x)]$
$[\![$more than three$]\!]$	$= \lambda D_{\langle d,t \rangle} . max(\lambda d . D(d)) > 3$
$[\![$more than three students smiled$]\!]$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) > 3$
$[\![$less than three$]\!]$	$= \lambda D_{\langle d,t \rangle} . max(\lambda d . D(d)) < 3$
$[\![$less than three students smiled$]\!]$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) < 3$
$[\![$at most three$]\!]$	$= \lambda D_{\langle d,t \rangle} . \neg \exists d[d > 3 \wedge D(d)]$
$[\![$at most three students smiled$]\!]$	$= \neg \exists d[d > 3 \wedge \exists x[|x| = d \wedge \text{students}(x) \wedge \text{smiled}(x)]$
$[\![$at least three$]\!]$	$= \lambda D_{\langle d,t \rangle} . D(3)$
$[\![$at least three students smiled$]\!]$	$= \exists x[|x| = 3 \wedge \text{students}(x) \wedge \text{smiled}(x)]$

For the most part this theory focuses on CMs and their connection to comparative constructions more generally. As such it does not cover scalar implicatures, ignorance, acceptability in downward-entailing environments, predicative uses. Entailments are captured in full, but compositionality is captured only in part and only for CMs, by giving *more/less* meanings obtained via a maximality operator and the $>$ / $<$ relations, similar to the meanings given to comparative constructions elsewhere in the literature (e.g., in Heim 2000); constituency is captured, but the modified numeral has to move, which raises the question as to why we can so easily conjoin a modified numeral DP with another DP: *more than three students and every professor*.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	?	?	?	✓/?	?	✓/?

It would seem that our theory manages to meet Hackl's compositionality desideratum for GQT without abandoning the convenient way in which GQT gets the lower- and upper-bounding entailments of modified numerals, namely, through direct reference to the cardinality of the intersection of P and Q. Hackl (2000:158) however claims that the simpler, GQT way of getting the entailments of modified numerals is not adequate and that 'insurmountable' evidence against it and in favor of truth conditions for CMs based on *max* and $\exists x$ comes from scope splitting, namely, from the fact that the reading of (99) where the CM takes wide scope with respect to *require* must be written as $max \ldots \square \ldots \exists x$, where *require* comes in between the *max* piece of CMs (encoded by the comparative morpheme) and the $\exists x$ piece of CMs (encoded by the silent *many*):

(99)   John is required to read fewer than 5 papers.
        $max\{d : \square \exists x[|x| = d \wedge \ldots]\} < 5$
        'The maximum degree d such that in every world there is an x which is a plurality of books counting d-many atoms and John reads x.'

However, to the best of my understanding this only shows that Hackl's meaning for CMs relying on *max* and $\exists x$ leads to a scope splitting configuration when the CM takes wide scope with respect to some other operator. It does not prove that the GQT approach to entailments, without reference to *max* or $\exists x$, is wrong. In fact, in GQT, or in our theory, *more/less than n P*, or for that matter also *at most/least n P*, are plain generalized quantifiers, which means that we can very easily move them for scope interactions as we in fact already did for various examples discussed in §2.2.2 or §2.2.3. Unless we find examples where Hackl's $max + \exists x$ approach would yield readings that our GQT-inspired approach can't replicate, our approach still stands, and is also more economical.

In contrast, Hackl's use of both *max* and $\exists x$ leads to problems. The first is that we might expect scope interactions between the two, which in the $\exists x \ldots max$ configuration would lead to van Benthem's problem for *less than* (i.e., lower-instead of upper-bounding truth conditions). Hackl averts this by hard-wiring $\exists x$ into the meaning of *many*. The second one is the fact that a maximality operator is problematic in cases where we have no defined maximum. More concretely, in the example below (borrowed from Buccola and Spector 2016) the maximality operator should be undefined because the degree predicate that it operates on does not in fact have a maximum, because if, say, 7 eggs are sufficient, any number higher than 7 will also be:

(100)   Less than ten eggs are sufficient to make an omelet for all these people.
        $max(\lambda n . \exists x[|x| = n \wedge \text{eggs}(x) \wedge \text{sufficient}\ldots(x)]) < 10$

Hackl notes this problem (Hackl 2000:163, fn127) but does not offer a solution. Note that this problem does not arise with our GQT-style derived maximality since we would simply be saying that the intersection of things that are eggs and of things that are sufficient to make an omelet counts less than 10 atoms:

(101) $|\ [\![\text{eggs}]\!] \cap [\![\text{sufficient}\ldots]\!]\ | \in \overline{[\![\text{little}]\!]\ (10)}$, i.e., $|\ [\![\text{eggs}]\!] \cap [\![\text{sufficient}\ldots]\!]\ | \in \{\ldots, 8, 9\}$

A number of theories after Hackl (2000) (e.g., Nouwen 2010, Kennedy 2015, Buccola and Spector 2016, a.o.) use *max* and $\exists x$. To the extent that those theories use these notions in the same way as Hackl, all the observations made here apply in those cases also.

3.4   Fox and Hackl (2006)

Fox and Hackl (2006) propose that the measurement scales needed for natural language semantics are always infinitely dense: For any n and $n + \epsilon$ there is a degree $n + \delta$ such that $n < n + \delta < n + \epsilon$. They show that this helps us capture the missing scalar implicatures of CMs in unembedded environments. The idea is that, given an infinitely dense scale, the scalar implicatures of an unembedded CM utterance would contradict the assertion. Let's see this more closely. *John weighs more than 120 pounds* is true iff John weighs some amount of pounds greater than 120, e.g., $120 + \epsilon$. Now, if *John weighs* $120 + \epsilon$ *pounds* is true, then *John weighs more than* $120 + \epsilon/2$ *pounds* must also be true because $120 + \epsilon > 120 + \epsilon/2$. However, if the scale is universally dense, then *John weighs more than* $120 + \epsilon/2$ *pounds* is one of the stronger alternatives to the assertion, so the scalar implicature process would negate it, which would contradict the entailment we showed earlier. Fox and Hackl argue that this is why CMs do not give rise to scalar implicatures in unembedded contexts.

The same reasoning also captures the fact that the scalar implicatures of CMs reappear under universal modals.

However, this proposal also derives a number of undesirable results. As Mayr (2013) shows, it systematically makes the opposite sort of predictions for CMs and SMs, failing to capture the fact that their patterns with respect to scalar implicatures are identical. For example, while, as mentioned, it predicts the scalar implicatures of CMs to be missing in unembedded environments and to reappear when the CM is embedded under a universal modal, it predicts the scalar implicatures of SMs to be *present* in unembedded environments and missing when embedded under a universal modal. Additionally, it wrongly predicts that CMs but not SMs should give rise to scalar implicatures when embedded under negation, whereas neither of them do. In a nutshell, this account both overgenerates and undergenerates, and in general fails to capture the fact that the patterns of CMs and SMs with respect to scalar implicatures are entirely parallel.

The account does not discuss the other phenomena of interest to us.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
?	✓/✗	?	?	?	?	?

Although aimed to capture the scalar implicature pattern of CMs, the universal density of measurement assumption does not thus yield all the right results, neither for CMs, nor for numerals more generally. Is it however a plausible or possible idea in principle? Our account does not rule it out that the scale could sometimes be infinitely dense, or even that such an assumption could sometimes be needed to handle natural language phenomena. The point is simply that that is usually not the case, and that in out-of-the-blue contexts with BNs, CMs, and SMs the type and granularity of the scale seems to be determined implicitly by what we are counting. For example, when we count individuals (which was essentially the case in all our examples so far) the implicit scale is typically a scale of positive numbers with granularity 1. In other types of settings (e.g., in a lab, or when talking about the temperature, or in the context of a game where each scores come in multiples of three, etc.) the implicit range and granularity of the scale could very well be quite different. Our account allows for all these possibilities.

3.5   Geurts & Nouwen (2007)

Geurts and Nouwen (2007) treat *n P* and *more/less than n P* as predicates where the former is factored into the meaning of the latter. The latter are derived as follows: *more/less than* operate on (totally, $<$)

ordered sets of alternatives and when applied to a predicative argument α they return another predicate - the set of individuals x such that there is an alternative to α that stands in a particular ordering relation to α and which is true of x. All these predicates can undergo existential ($\exists$) or universal ($\forall$) closure (which are defined as covert existential/universal quantifiers, (102)), which can apply freely (with undesirable readings assumed to be ruled out based on triviality or implausibility).

(102) a. $\exists A = \lambda P . \exists x [A(x) \wedge P(x)]$, where A is of type $\langle e, t \rangle$
       b. $\forall A = \lambda P . \forall x [P(x) \rightarrow A(x)]$, where A is of type $\langle e, t \rangle$ (from De Swart 2001)

Unlike Krifka, Geurts and Nouwen don't just distinguish between bare and modified numerals, but also between CMs and SMs, arguing that the latter are modal in nature. Like comparative modifiers, *at most/least* operate on (partially, $\leq$) ordered sets of alternatives; however, when applied to a predicative argument α they return a generalized quantifier that applied to a predicate with a verbal meaning P returns true iff it is possible that $[\exists \alpha](P)$ and there is no higher-ranking alternative β such that it is possible that $[\exists \beta](P)$ (*at most*) or it is necessary that $[\exists \alpha](P)$ and there is a higher-ranking alternative β such that it is possible that $[\exists \beta](P)$ (*at least*).

(103)   ('$\alpha \triangleright \beta$' stands for '$\alpha$ outranks β')

$\llbracket \text{three students} \rrbracket$ $\quad = \lambda x . [|x| = 3 \wedge \text{students}(x)]$

$\llbracket \text{more than } \alpha \rrbracket$ $\quad = \lambda x . \exists \beta [\beta \triangleright \alpha \wedge \beta(x)]$
$\llbracket \text{more than three students} \rrbracket$ $\quad = \lambda x . \exists \beta [\beta \triangleright \llbracket \text{three students} \rrbracket \wedge \beta(x)]$
$\quad = \lambda x . [|x| > 3 \wedge \text{students}(x)]$

$\llbracket \text{less than } \alpha \rrbracket$ $\quad = \lambda x . \exists \beta [\alpha \triangleright \beta \wedge \beta(x)]$
$\llbracket \text{less than three students} \rrbracket$ $\quad = \lambda x . \exists \beta [\text{three students} \triangleright \beta \wedge \beta(x)]$
$\quad = \lambda x . [|x| < 3 \wedge \text{students}(x)]$

$\llbracket \text{at most } \alpha \rrbracket$ $\quad = \lambda P_{\langle e,t \rangle} . \Diamond [\exists \alpha](P) \wedge \neg \exists \beta [\beta \triangleright \alpha \wedge \Diamond [\exists \beta](P)]$
$\llbracket \text{at most three students} \rrbracket$ $\quad = \lambda P_{\langle e,t \rangle} . \Diamond \exists x [|x| = 3 \wedge \text{st}(x) \wedge P(x)] \wedge \neg \Diamond \exists x [|x| > 4 \wedge \text{st}(x) \wedge P(x)]$

$\llbracket \text{at least } \alpha \rrbracket$ $\quad = \lambda P_{\langle e,t \rangle} . \Box [[\exists \alpha](P) \wedge \exists \beta [\beta \triangleright \alpha \wedge \Diamond [\exists \beta](P)]$
$\llbracket \text{at least three students} \rrbracket$ $\quad = \lambda P_{\langle e,t \rangle} . \Box \exists x [|x| = 3 \wedge \text{st}(x) \wedge P(x)] \wedge \Diamond \exists x [|x| > 4 \wedge \text{st}(x) \wedge P(x)]$

This theory derives the bounding entailments through existential closure (*three*, *more than three*) / universal closure (*less than*) / by explicitly negating that there are stronger alternatives that are possible *at most*; the missing scalar implicatures of CMs and SMs – by saying that the alternatives are already exploited by the modifiers; the epistemic effect of SMs – by hardwiring epistemic modals in their meaning; and the infelicity of SMs under negation or other types of downward-entailing environments – by arguing that epistemic modals can't embed. To capture the readings of SMs embedded under modals they propose a new rule of modal concord.

This theory is however unable to derive any of the other scalar implicature patterns, or to explain why ignorance is present in CMs also and why sometimes it is optional for SMs too, or why SMs are in fact acceptable in the antecedent of a conditional or the restriction of a universal (in those cases, in fact, as the authors themselves comment, the truth conditions yield the wrong meaning).

Finally, the theory is partly compositional (*three* means the same everywhere), it gets predicative meanings for BNs and CMs but not SMs (as they show, $\exists$ has to happen below the modal, which forces *at most/least n P* to end up with a generalized quantifier meaning), and it respects constituency, but only partly, just like Krifka's account.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓/✗	✓/✗	✓/✗	✓/?	✓/?	✓/?

Note that while this theory meets the entailments desideratum, it does so potentially less economically than in our account. For example, upper-bounding entailments are derived differently for *less than* and *at most*. There are also problems coming from the fact that SMs are modal. One is the fact that getting the right meaning for SMs under a modal requires further non-compositional mechanisms such as modal concord. Another is the fact that an epistemic modal meaning makes SMs intensional, whereas as Cohen and Krifka (2011) show, SMs are extensional. Also see Mayr (2013) for further criticism on how this theory yields the wrong truth conditions for sentences where a universally quantified noun phrase takes wide scope with respect to the SM.

3.6 Büring (2008)

Büring (2008) proposes that *at least q* yields a disjunction where the first disjunct is given by *q* minus the union of all the alternatives ranked higher than *q* (essentially an exhaustified *q*) and the second disjunct is given by the union of all the alternatives ranked strictly higher than *q*. In plainer terms, *at least n* is equal to the disjunction of *exactly n* and *more than n*.

(104) $[\![\text{at least q}]\!] = [[\![q]\!] - \bigcup(\text{ABOVE}(q))] \vee \bigcup(\text{ABOVE}(q))$

This theory gets the bounding entailment of *at least*, its missing scalar implicature in unembedded environments, its ignorance implicatures in unembedded environments (through pragmatic reasoning negating the stronger domain alternatives given by the individual disjuncts: $\neg Bel_S(\text{exactly n})$ and $\neg Bel_S(\text{more than n})$), and its implicatures when embedded under a universal modal (via the same reasoning as the one used to derive ignorance implicatures).

Because no scalar implicatures are active, this theory does not get the attested direct implicatures of *at least* in contexts with granularity different from 1 or its indirect scalar implicatures. The account does not address the (in)felicity of *at least* in downward-entailing environments, compositionality, predicative uses, or constituency. It also doesn't address BNs and CMs.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	✓/✗	✓/?	?	?	?	?

Büring's account uses Krifka's way of neutralizing the scalar alternatives of SMs, but by saying that SMs have an overt disjunctive form he also ensures that SMs still have some active alternatives that can yield implicatures. These are then used to derive implicatures (ignorance implicatures in unembedded contexts and scalar-like implicatures when *at least* is under a universal operator). An important contribution of this account is thus the intuition that at least part of the behavior of SMs must be derived from what are essentially domain alternatives. We will see this intuition at work again in Kennedy (2015) (§3.11) and Spector (2015) (§3.12). One challenge for all these accounts is to explain in what way the domain alternatives are obtained. For example, as Coppock and Brochhagen (2013) point out, while

the intuition that SMs are in some sense disjunctive seems on the right track, it is not obvious that they should contain an overt disjunction in their truth conditions. In our account this disjunctive flavor is derived.

3.7 Nouwen (2010)

Nouwen (2010) starts from Hackl (2000)'s meanings for CMs, which involve a maximality operator that says that the maximal degree n such that there is a plurality with cardinality n is less/greater than the number that is being modified by the CM. In addition to Hackl's silent existential quantifier ($\exists$) (encoded in the meaning of the degree predicate), Nouwen also stipulates a second one that asserts unique existence ($\exists!$) and which he uses to derive a simplex *exactly* meaning for the bare numerals (meaning (2) below) alongside the traditional *at least* meaning (meaning (1) below). He then proposes that SMs have a meaning on the surface similar to CMs, only crucially different in that SMs pick out a minimum (*at least*) or a maximum (*at most*) value. Moreover, he proposes that whenever the meaning of the SM is contradictory [16] or equivalent to that of the exact meaning (meaning (2)) of the bare numeral, the LF produced by the SM is banned;[17] it can be rescued by the insertion of a silent existential epistemic modal under max or min:

(105)

$[\![$more than three students smiled$]\!]$	$= max_n(\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) > 3$
$[\![$less than three students smiled$]\!]$	$= max_n(\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) < 3$
$[\![$three students smiled$]\!]$	$(1) = \exists x[\text{students}(x) \land \text{smiled}(x)]$
	$(2) = \exists!x[\text{students}(x) \land \text{smiled}(x)]$
$[\![$at most three students smiled$]\!]$	$= max_n(\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) = 3$
if $\bot$ or blocked by *three*-(2)	$= max_n(\Diamond\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) = 3$
$[\![$at least three students smiled$]\!]$	$= min_n(\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) = 3$
if $\bot$ or blocked by *three*-(2)	$= min_n(\Diamond\exists(!)(x)[\#(x) = n \land \text{students}(x) \land \text{smiled}(x)]) = 3$

This theory derives the bounding entailment as well as the optional scalar inference that leads to an exactly meaning of BNs by making them lexically ambiguous between a lower- and a dual-bounded meaning. It derives the bounding entailments of CMs by saying that the maximum degree that satisifes the degree property is more/less than some value, and the bounding entailments of SMs by making them maxima/minima operators. It captures the epistemic flavor of SMs by ensuring that their meaning will always end up containing an epistemic modal. To capture certain readings of SMs under universal modals it invokes an additional rule to the effect that universal modal operators must sometimes be interpreted with an existential force.

While this theory is able to derive scalar implicature-like meanings for BNs, it is unable to derive their indirect implicatures, e.g., in the antecedent of conditionals, where the result was not an 'ex-

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[16]Which happens, for example, every time min combines with $\exists$, e.g., $[\![$at least three$]\!] = min\,(\exists\ldots) > 3\,(\bot)$ because the minimum value for which there exists such a property will always be 1 (at least, for distributive predicate).

[17]As it turns out, this LF is banned in all the cases that Nouwen discusses, so the meaning that he starts out with for SM is a completely abstract meaning.

actly' meaning but rather an inference based on another numeral. It does not directly address scalar-implicature-like meanings for modified numerals but could potentially derive some of them via various mechanisms (e.g., as Nouwen suggests, additional rules for how an SMs must be interpreted under a universal modal). While it gets some of the epistemic flavor of SMs, it however does not derive speaker *ignorance*, but only says what values the speaker considers to be possible. It does not address the infelicity of SMs under negation but we could imagine that it could be explained by the fact that an epistemic modal can't embed. However, this leaves unexplained the fact that SMs *can* embed in the antecedent of a conditional or the restriction of a universal. Also, although an SM is degraded under negation, we do however get a clear sense of what its truth conditions would be, namely, *Alice doesn't have at least three diamonds* means that she has less than three; however, on this theory the predicted meaning would be that the it is not the case that the maximum degree d such that there is a world where Alice has d-diamonds possible number is 3, that is, the utterance is predicted to be true if Alice has any number of diamonds so long as it is not 3. The theory does not address compositionality beyond borrowing Hackl's meaning for CMs (which rely on certain assumptions about the comparative morpheme). It also does not address predicative uses or constituency except potentially for BNs (which are presumably given a predicative meaning).

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓/✗	✓/✗	✓/?	?	✓/?	✓/?

This theory gets the entailments but the way it achieves this creates further problems. For example, lexical ambiguity for BNs ensures they entail a lower bound and also gets their optional upper bound, but without alternatives this setup can't derive their indirect implicatures. Then, the use of *max* in *more than* is vacuous (a simple existential meaning would have done the job; see §3.3). Moreover, the fact that epistemic flavor of SMs is captured via last resort insertion of a possibility epistemic modal, with no further restrictions on its insertion other than competition with a bare numeral, leads to problems with embedding; see Kennedy (2015) for further criticism. Finally, saying that SMs yield equality between a maximum/minimum value and the numeral wrongly makes them non-monotonic (*at most* is downward-entailing and licenses NPIs; see Chierchia and McConnell-Ginet 2000, Schwarz et al. 2012).

3.8   Mayr (2013)

Mayr (2013) focuses on CMs and SMs and tries to account for the fact that they don't give rise to scalar implicatures in unembedded contexts or under negation but do when embedded under a universal modal. He proposes that the alternatives of a modified numeral are the alternatives that result from the cross-product of the alternatives to the modifier (*more/less than* are each other's alternatives, and so are *at most/least*) with the alternatives to the numeral (the traditional Horn set). For example, *more than 3* doesn't have as alternatives just {…, *more than 4, more than 5*, …}, but also {…, *less than 3, less than 4*, …}. Using a variant of the grammatical theory of scalar implicatures (essentially, the contradiction-free system of Chierchia et al. 2012), he shows that from these alternatives one can derive all the implicature patterns just mentioned.

This theory assumes existential meanings for *more than n P Q* and *at least n P Q* and doesn't say anything about the upper-bounding entailments of *less than* and *at most*. The alternative set ensures that no scalar alternatives can arise for either CMs and SMs in unembedded contexts, which means that the theory is unable to handle the implicatures of CMs and SMs in unembedded environments with non-default scale. (It is also unclear what this new alternative set predicts for indirect implicatures.) The account

moreover doesn't address ignorance implicatures, acceptability in downward-entailing environments (although the author does note that SMs are degraded under negation), the internal compositionality of CMs and SMs, their predicative uses, or constituency.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	✓/✗	?	?	?	?	?

An important contribution of this paper is the observation that CMs and SMs behave in entirely parallel ways with respect to scalar implicatures. Our theory preserves this insight although we derive scalar implicatures from traditional scalar alternatives and are thus also able to handle the cases where we do see CMs and SMs giving rise to scalar implicatures (e.g., in contexts with coarse granularity scale).

3.9   Cohen and Krifka (2011, 2014)

Cohen and Krifka (2011, 2014) focus on just SMs, and especially on their infelicity in downward-entailing environments. They propose that for these items falsity follows *semantically* (by saying *John petted at least three rabits* the speaker denies that John petted zero, one, or two rabbits) and whereas truth follows *pragmatically* (for all the values that the speaker did not deny, the hearer concludes by implicature that the speaker GRANTS that John petted that number of rabbits, i.e., that he petted three, four, five, ..., rabbits.) Thus, to interpret SMs, one must compute an implicature. But, since scalar implicatures tend to be cancelled in downward-entailing environments, this explains why SMs are infelicitous under negation.

Of course, this raises the issue of why SMs are felicitous in other downward-entailing environments such as the antecedent of conditionals or the restriction of universals. Following Kay (1992), Cohen and Krifka suggest that this is a different meaning of SMs, a so-called 'evaluative' sense that could give us truth-conditions in downward-entailing environments without the need to compute an implicature.

However, then we would expect such evaluative uses to be able to make SMs felicitous in the scope of negation also. However, an SM in a negative declarative seems to be bad regardless of whether it is used evaluatively and also (another potential factor) regardless of whether the property it modifies is assumed to be positive (being centrally located) or negative (being far away):[18]

(106) a. ??This hotel isn't at least centrally located.
       b. ??This hotel isn't at least far away.

Cohen and Krifka admit that this is something they cannot explain.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	?	?	✓/✗	?	?	?

This theory does get the entailments of SMs and also captures part (though not all) of their patterns of acceptability in downward-entailing environments. These results are however obtained via a significant amount of unorthodox machinery the consequences of which are not clear. (See, for example, Coppock and Brochhagen 2013 for some criticism of the idea of falsity conditions following semantically

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[18]Mihoc and Davidson (2017) show that the acceptability of SMs in antecedents or restrictors may indeed be sensitive to the positive or negative valence of the continuation, but in a way that doesn't quite match the predictions of Cohen and Krifka's account for the evaluative uses of SMs either. In this paper I do not address that but in principle I believe it is an independent complicating factor rather than a fundamental issue.

and truth conditions following pragmatically.) In contrast, our proposal is anchored in mechanisms that are much better understood and independently justified, and also captures both the unacceptability of SMs in the scope of negation and their acceptability in the antecedent of a conditional or the restriction of a universal.

3.10 Coppock and Brochhagen (2013)

Coppock and Brochhagen (2013) also primarily focus on SMs. They propose that *at most/least n P Q* are Inquisitive Semantics-style propositions (= sets of sets of *possibilities*, = sets of sets of worlds, = sets of classical propositions). Informally, *at least n P Q* is the set containing the prejacent and its higher-ranked scalar alternatives and *at most n P Q* is the set containing the prejacent and lower-ranked scalar alternatives. This gets the right bounding entailments of SMs. Then, the absence of scalar implicatures of SMs is derived via an Inquisitive Semantics notion of exhaustification which is defined such that no possibilities end up being removed from the meaning of the SM-utterance and thus no scalar implicatures are derived. The ignorance implicatures of SMs are obtained via a Inquisitive Semantics pragmatic mechanism of interactive sincerity (where *interactive* describes a proposition that contains more than one possibility). (CMs are regarded as non-interactive, so they don't give rise to ignorance implicatures.) Embedding under modals is ensured compositionally with suitable Inquisitive-Semantic meanings for the modals, and implicatures in those environments are obtained via scope interactions and use of the maxim of interactive sincerity mentioned earlier.

The theory does not capture those cases where SMs (and CMs) did give rise to scalar implicatures. It fails to capture the fact that CMs can give rise to ignorance too. The authors admit that they do not have an explanation for why SMs are bad under negation. The theory does not address compositionality, predicative uses, or constituency (but the superlative modifiers operate at a sentential level, so they do not form a constituent with the numeral).

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	✓/✗	✓/✗	?	?	?	?

An important feature of this theory is the fact that it tries to capture the similarity between SMs and disjunction at a formal level. This is done with the help of Inquisitive Semantics, where both SMs and disjunction are *interactive* propositions (sets of sets of worlds), that is, propositions that contain multiple possibilities (sets of worlds – essentially, the prejacent + alternatives). However, on this theory disjunction is given essentially domain alternatives, whereas SMs are given essentially scalar alternatives.[19] In contrast, our account captures the similarity between disjunction and SMs even further: they both have both scalar and domain alternatives.

3.11 Kennedy (2013, 2015)

Kennedy (2013, 2015) assumes parallel meanings for both bare and modified numerals – they are generalized quantifiers over degrees, (107), which are generated as sisters to Hackl (2000)'s silent

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[19]Disjunction is treated as interactive-inquisitive, i.e., it contains two maximal possibilities, and SMs are defined as interactive-*attentive*, i.e., they contain (at least) one non-maximal possibility. Note that the status of attentivity is unclear in the current incarnation of Inquisitive Semantics (e.g., Ciardelli et al. 2013) since a fundamental assumption is that propositions are *downward-closed* sets of possibilities, which raises the question of how to figure out what the relevant non-maximal possibilities are.

many, (108), but can't be interpreted in situ due to type mismatch (see tree in §3.3) so they must move, giving rise to meanings as in (109).

(107) $\llbracket$three / more/less than three / at most/least three $\rrbracket$
$= \lambda D_{\langle d,t \rangle} . max(\lambda n . \mid D(n)) = / > / < / \leq / \geq 3$

(108) $\llbracket many \rrbracket = \lambda n_d . \lambda P_{\langle e,t \rangle} . \lambda Q_{\langle e,t \rangle} . \exists x[|x| = n \wedge P(x) \wedge Q(x)]$

(109)

$\llbracket$three students smiled$\rrbracket$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) = 3$
$\llbracket$more than three students smiled$\rrbracket$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) > 3$
$\llbracket$less than three students smiled$\rrbracket$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) < 3$
$\llbracket$at most three students smiled$\rrbracket$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) \leq 3$
$\llbracket$at least three students smiled$\rrbracket$	$= max(\lambda n . \exists x[|x| = n \wedge \text{students}(x) \wedge \text{smiled}(x)]) \geq 3$

This theory captures the upper-bounding inference of BNs as an entailment, BNs being given a dual-bounded, 'exact' meaning. The 'at least' meaning in unembedded environments is derived by typeshifting via Partee (1987)'s BE and *iota* to obtain a plain degree meaning which goes on to saturate the degree position of *many* directly, giving rise to an 'at least' meaning.

(110) a. $\llbracket BE \rrbracket = \lambda Q_{\langle\langle a,t \rangle,t \rangle} . \lambda x_\alpha . Q(\lambda y_\alpha . y = x)$
          (i) $\llbracket BE \rrbracket (\llbracket three \rrbracket)$
               $= \lambda x_d . x = 3$
       b. $\llbracket iota \rrbracket = \lambda P_{\langle a,t \rangle} . \iota z_\alpha[P(z)]$
          (i) $\llbracket iota \rrbracket (\lambda x_d . x = 3)$
               $= 3$
       c. $\llbracket many \rrbracket (3) = \lambda P ._{\langle e,t \rangle} \lambda Q ._{\langle e,t \rangle} \exists y[P(y) \wedge Q(y) \wedge \#(y) = 3]$

The bounding entailments of CMs and SMs are also captured via the maximality operator, which ensures that *less than* and *at most* entail an upper bound.

Kennedy proposes that, instead of having scalar alternatives, BNs, CMs, and SMs with the meanings in (109) are each other's alternatives. Using neo-Gricean pragmatic mechanisms of calculating primary and secondary implicatures (Sauerland 2004) and capitalizing on the fact that in the proposed set of alternatives only SMs have stronger alternatives (at least in upward-entailing environments), it also derives ignorance and no scalar implicatures for SMs in unembedded environments, but reappearing scalar implicatures when the SM is embedded under a universal modal.

(111) (below, 'max' abbreviates 'max$(\lambda n . \exists x[\#(x) = n \wedge \text{students}(x) \wedge \text{smiled}(x)])$')
       At least three students smiled.
       Alternatives: max $= 3$, max > 3, max < 3, max ≤ 3
       Primary implicature: $\{\neg K(\text{max} = 3), \neg K(\text{max} > 3)\}$ (ignorance implicatures)
       Secondary implicature: None. (no scalar implicatures)
       a. $K(\neg(\text{max} = 3)) \wedge (\text{max} \geq 3) \wedge \neg K(\text{max} > 3) \Rightarrow \bot$
       b. $K(\neg(\text{max} > 3)) \wedge (\text{max} \geq 3) \wedge \neg K(\text{max} = 3) \Rightarrow \bot$

Although it gets both the 'at least' and the 'exactly' meanings of BNs, because it does not allow for scalar alternatives, the account cannot capture the indirect implicatures of BNs (the indirect implicature of

a numeral n is not about the 'at least' or 'exactly' meaning of n but rather about $n-1$). For the same reason it also cannot capture the direct scalar implicatures of CMs or SMs in unembedded contexts with a coarser granularity of the scale or their indirect scalar implicatures. Because in the proposed alternative set of alternatives SMs have stronger alternatives (at least in certain environments) but CMs don't, the account moreover does not capture the fact that SMs and CMs show similar patterns with respect to the absence and reappearance of scalar implicatures, or to ignorance. The theory doesn't address acceptability in downward-entailing environments (but in principle no difference is predicted between CMs and SMs), compositionality (except for the similarity to comparatives inherited from assuming Hackl's meanings for CMs), predicative uses, or constituency (same discussion as in Hackl, §3.3).

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓	✓/✗	✓/✗	?	✓/?	?	✓/?

The theory gets the correct bounding entailments for *less than* and *at most* via the maximality operator. However, the use of the maximality operator is vacuous for for *more than* and *at least* (see §3.3). Our theory derives all these bounding entailments more economically.

This theory also gets the ignorance implicatures of SMs in a similar way to ours. However note that the alternatives are completely stipulative, and the attempt to make it less so by saying that BNs, CMs, and SMs are one another's alternatives backfires for CMs because it makes them unable to give rise to any implicatures. In contrast on our account the alternatives of CMs and SMs fall out naturally from their truth conditions.

3.12 Spector (2015)

Our account is a direct expansion and refinement of Spector (2015). It is therefore important to see which features of Spector's account we preserved and which we improved upon.

First, let's use our usual measuring stick to get a sense of the scope and tools of this theory.

Spector (2015) starts from the infelicity of SMs under negation. He notes that SMs behave the same as Hurford disjunction or positive polarity item (PPI)-like disjunctions more generally:[20] they don't give rise to scalar implicatures (Spector doesn't consider the coarse granularity scale contexts where they do), they have obligatory ignorance implicatures in unembedded contexts, and they resist embedding in the scope of negation. He proposes that SMs must be parsed as Hurford disjunctions, with the first disjunct exhaustified: *at least three = exh(at least three) or at least four = exactly three or at least four* and *at most three = exh(at most three) or at most two = exactly three or at most two*. He furthermore argues that the disjunctive alternatives of SMs are obligatory. Using a variant of the grammatical theory of scalar implicatures (the contradiction-free system of Chierchia et al. 2012 with the exhaustivity operator *exh*, Innocent Exclusion, Meyer 2013's silent, matrix-level, universal epistemic modal K, and an economy condition on vacuous exhaustification), he derives all the above-mentioned patterns. (These are essentially the same results that we got from $O^{PS}_{SubDomAlts}$ in §2.2.3-§2.2.4, minus our discussion of SMs in presuppositional downward-entailing environments.)

This account furthermore gets the scalar implicature patterns of SMs when SMs are embedded under a universal operator, their ignorance implicatures, and infelicity under negation.

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[20]Hurford disjunction is a type of disjunction where one disjunct entails the other and where, to avoid violating Hurford's constraint against such disjunctions, one disjunct must be interpreted exhaustively. It is often discussed as evidence in favor of embedded implicatures (see, e.g., Chierchia et al. 2009).

However, it does not capture the scalar implicatures of SMs in unembedded contexts with coarse granularity scale, or their indirect scalar implicatures. It does not address the acceptability of SMs in the antecedent of a conditional or the restriction of a universal (although in a different paper, Spector 2014, the author does mention the fact that the solution probably lies with the effect of a presupposition on exhaustification – that is, the solution that we implemented). Moreover, the account does not discuss the entailments and implicatures of BNs (although from Spector 2013 we may guess that the author would favor the same account of the BNs as ours) or CMs, nor does it address compositionality, predicative uses, or constituency.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	✓/✗	✓/?	✓/?	?	?	?

While Spector's account of the ignorance implicatures of SMs roughly parallels the pragmatic account of Kennedy (minus the way the alternatives are obtained and the exhaustification system), his is the first pragmatic account that explicitly tries to explain their obligatoriness and moreover uses the same mechanism to cash out the unacceptability of SMs under negation. However, these results are obtained at the cost of making certain stipulations about the form and status of the alternatives of SMs. The question is then: Why do these stipulations make sense for SMs, and what do they entail for BNs and CMs?

More concretely, Spector proposes that SMs have what are essentially symmetric domain alternatives (e.g., ALT(*at least three*) = {*exh*(*at least three*), *at least four*} = {*exactly three, at least four*}) and shows that from these alternatives one can derive certain implicatures, which, if obligatory, would produce no scalar implicatures and obligatory ignorance inplicatures in unembedded contexts, and infelicity under negation. He does not discuss CMs, but if we make similar assumptions about their alternatives (e.g., ALT(*less than three*) = {*exh*(*less than three*), *less than two*} = {*exactly two, less than two*}) but the implicatures are optional, we get no scalar implicatures and optional ignorance in unembedded contexts, and acceptability under negation. However, if we make the same assumptions for BNs (i.e., ALT(*three* = at least three) = {*exh*(*at least three*), *at least four*} = {*exactly three, at least four*}), we predictably get the same results as for CMs – no scalar implicatures and ignorance implicatures – which for BNs are however wrong; to get the signature scalar implicatures of BNs right, BNs crucially need to have *scalar* alternatives. So then the following overarching questions emerge: Why should SMs and CMs have only two alternatives but BNs – arbitrarily many? Why should SMs and CMs have only domain alternatives but BNs only scalar alternatives? And why obligatory exhaustification for the domain alternatives of SMs but not for those of CMs?

While our account directly borrows Spector's reasoning for SMs, it also answers all of these questions: BNs, CMs, and SMs all have an arbitrary number of scalar alternatives (which helps us get the patterns that Spector's account couldn't get), and CMs and SMs – an arbitrary number of subdomain alternatives (limited only by the size of the domain and the granularity of the scale). CMs and SMs have subdomain alternatives because they rely in their composition on a set of degrees based on the numeral (and built via *much*/*little* and [comp]/[sup]), which is not the case for BNs. And, finally, the subdomain alternatives of SMs are obligatory because the presupposition introduced by the superlative morpheme forces the domain to have a certain size, which ultimately forces SMs to always have subdomain alternatives.

The fact that, as Spector shows, the same solution that gives us the signature behavior for SMs also gives us the behavior of other positive polarity items more generally, (and in fact with similar tools as those used to derive the behavior of negative polarity items; see Chierchia 2013) is a welcome bonus: we

now have not just a theory of bare and modified numerals that captures their connections to scalar items, adjectives, and comparative/superlative modification, but also a theory that captures their connection to the polarity system.

3.13   Buccola and Spector (2016)

Buccola and Spector (2016) focus on the meaning of *less than n P Q* and argue, against what we have been assuming so far, that it does not always entail an upper bound, but rather that it has an upper-bounded meaning when it combines with distributive predicates and a lower-bounded meaning when it combines with a collective predicate. Specifically the claim is that (112) below entails that there is no group x such that x has more than four members and they are students who smiled, and is also consistent with no student having smiled at all, while (113) on its collective reading (as primed by the context below) does not carry such an upper-bounding entailment and also carries an existential entailment to the effect that someone did lift the piano.

(112)   Less than four students smiled.

(113)   *Context: A group of boys want to see how many of them it takes to lift the piano. Specifically, they want to know if less than four of them can do it. Different groups take turns trying to lift the piano. In the end, one group of five, one group of four, and one particularly strong group of three boys manage to lift it.*
         Less than four students lifted the piano.[21]

They argue thus that any theory of *less than* needs to be able to generate both meanings, each in the right context only. They set out to do this. To begin with, they assume that the upper-bounded meaning is produced through the action of a maximality operator (as in Hackl 2000 or Kennedy 2015) and the lower-bounded meaning through existential quantification over individuals, schematically as in (114) and (115).

(114) $max(\lambda m \,.\, \exists x[|x| = m \land P(x)]) < n$

(115) $\exists x[|x| < n \land P(x)]$

Then they play with different ways of encoding maximality (as part of the meaning of the comparative modifier, or as a separate operator) and various scope interactions between *max* and $\exists x$[22] in an attempt to derive the desired results. None of their four attempts however proves successful since each setup ends up overgenerating. As such, the conclusion of Buccola and Spector (2016) is essentially an open puzzle but, importantly, a puzzle that would mean that none of the current theories of *less than n P Q* (or, by extension, *at most n P Q*), including our own, is complete.

But what exactly is the extent of this puzzle? Note that the context in (113) that Buccola and Spector use to argue for a lower-bounded meaning for *less than n P Q* is a context where the lifting events happen sequentially: Three groups (5 students, 4 students, and 3 students) take turns lifting the piano. Now, if events happen in succession, then we should relativize the event/agent atom count to a time. In that

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[21]Buccola and Spector (2016) are a little inconsistent in how they list these examples. Sometimes they include the word *together* to bring out the collective meaning, other times they don't. At all times however they have in mind the collective reading.

[22]Note that, unlike in Hackl 2000 or Kennedy 2015, the existential quantifier in Buccola and Spector (2016) is not introduced via *many*, so scope interactions are expected.

case maintaining a maximal meaning for *less than* is easy: maximality is verified at the time when the lifting by a group of three people happened because at the time of that event it is true that the agent of the lifting event counted less than four atoms. The real test case is then the case where the same three lifting events occur at the same time. This test is complicated a little by the fact that the object in the verbal predicate comes with the definite article (*the piano*) and it is odd to have three groups lifting the same piano at the same time. To avoid this complication, let's replace the definite article with the indefinite, *Less than four students lifted a piano*, and consider again a scenario where three groups – one of five, one of four, one of three – are lifting different pianos (narrow scope for the indefinite) at the same time. In this context I don't think *Less than four students lifted a piano* can be uttered truthfully. This become more obvious when we add a concrete time phrase:

(116)   Between 2 and 3 pm less than four people lifted a piano.

Intuitively this is false if between 2 and 3 pm there were three lifting events, one by a group of five, one by a group of four, and one by a group of three.

My tentative conclusion is thus that the evidence used to argue that there are cases where *less than n P Q* does not entail an upper bound does not in fact prove that. It is possible that further data and/or experimental probing could change that. Until then, I believe our upper-bounded truth conditions for *less than* (and *at most*) can be maintained.

As for our usual checklist, Buccola and Spector did not directly engage with any of our other desiderata other than entailments of *less than*, but did in passing cover parts of the compositionality of CMs and the predicative meaning and constituency of BNs. For the sake of completion, we record those too below.

ent's	scal implic's	ig implic's	DE environ's	compositionality	predicative uses	constituency
✓/?	?	?	?	✓/?	✓/?	✓/?

3.14   Summary

There have been many attempts to account for the various patterns of BNs, CMs, and SMs with which we started out. As shown above and summarized in the table below, none manage to check off all our desiderata, or at least none manage to do so in full. As such, there is a genuine gap that this paper can fill.

theory	ent's	scal implic's	ig implic's	DE	comp	pred	const
Barwise and Cooper (1981)	✓	✓/?	✓/?	?	?	?	✓
Krifka (1999)	✓	✓/✗	?	?	✓/?	✓	✓/?
Hackl (2000)	✓	?	?	?	✓/?	?	✓/?
Fox and Hackl (2006)	?	✓/✗	?	?	?	?	?
Geurts and Nouwen (2007)	✓	✓/✗	✓/✗	✓/✗	✓/?	✓/?	✓/?
Büring (2008)	✓/?	✓/✗	✓/?	?	?	?	?
Nouwen (2010)	✓	✓/✗	✓/✗	✓/?	?	✓/?	✓/?
Mayr (2013)	✓/?	✓/✗	?	?	?	?	?
Cohen and Krifka (2011, 2014)	✓/?	?	?	✓/✗	?	?	?
Coppock and Brochhagen (2013)	✓/?	✓/✗	✓/✗	?	?	?	?
Kennedy (2015)	✓	✓/✗	✓/✗	?	✓/?	?	✓/?

Spector (2015)	✓/?	✓/✗	✓/?	✓/?	?	?	?
Buccola and Spector (2016)	✓/?	?	?	?	✓/?	✓/?	✓/?
THIS PAPER	✓	✓	✓	✓	✓	✓	✓

At this point it would also be useful to do an overview of the crucial choice points, and to what degree the choices we have made turned out to be crucial or advantageous.

Entailments   Many of the existing theories seem to check off the entailments column. However, they vary quite a bit in how they do so. The first point of variation regards **the meaning of BNs**. Nouwen (2010) and Kennedy (2015) cash out the optionality of the upper-bounding inference of BNs not as a scalar implicature but rather as lexical ambiguity of a numeral such as *three* between an 'at least three' and an 'exactly three' meaning. (There is also further variation in how these two lexical meanings are assumed to be connected.) Because they forgo scalar alternatives, such theories are however unable to derive the attested indirect scalar implicatures of BNs. In contrast, following Horn (1972), Barwise and Cooper (1981) or Krifka (1999) derive the upper-bounding inference of BNs as a scalar implicature. This strategy can derive the indirect scalar implicatures of BNs, and is essentially the strategy that we followed. By adding to it the grammatical theory of implicatures, we were able to derive all the attested meanings of BNs, including their 'exactly' meaning in embedded contexts. The second point of variation in the entailments column regards the way in which **the bounding entailments of modified numerals** are captured, especially the upper-bounding entailments of *less than* and *at most*. Krifka (1999) derives them by saying that *less than* and *at most* mark scalar alternatives based on numerals higher than the bound as false. This however required the new notion of *falsity conditions*. Nouwen (2010) derives these entailments for SMs by having *at most / at least* yield ('=') the possible maximum / minimum of a degree property. This however results in wrongly non-monotonic meanings. Hackl (2000) (for CMs), Nouwen (2010) (for CMs), Kennedy (2015) (for CMs and SMs), and Buccola and Spector (2016) (for CMs) get them by comparing ($>, <, \leq, \geq$) the maximum of a degree predicate to a numeral. The maximality operator that picks out the maximum degree of this degree predicate is however used vacuously in *more than* and *at least*, and it also creates problems in contexts where the degree predicate does not have a maximum (see §3.3). Hackl (2000) in fact does something slightly different for SMs – *at least* gets an existential, and *at most* – a negative existential meaning. This raises the question as to where the negative component comes from. Finally, Barwise and Cooper (1981) get them by direct comparison ($>, <, \leq, \geq$) of the cardinality of the intersection of *P* and *Q* to a numeral, while Coppock and Brochhagen (2013), focusing just on SMs, similarly get them by treating *at most / at least* essentially as disjunctions over the scalar alternatives ranked at most/least as highly as the assertion. Both these approaches get the entailments they aim to get without any vacuous piece and in a way that respects monotonicity, and Coppock and Brochhagen also make SMs entail a disjunctive meaning. Our semantics for CMs and SMs achieves all these things: our meanings say that the cardinality of the intersection of *P* and *Q* is a degree in a certain lower- or upper-bounded interval, which gives us lower- and upper-bounding entailments in a way that is economical (no vacuous piece), captures monotonicity, and also gives concrete form to the intuition that they denote options from a range.

Scalar implicatures   Most of the existing theories also check off the scalar implicatures column. However, they do so to varying degrees, and none completely. The reason for that is usually the same: In an attempt to capture the *missing* scalar implicatures of CMs or SMs or both, these theories are set up such that they are unable to generate any or all of their *attested* scalar implicatures. In contrast, our

approach, which assumes that CMs and SMs do have scalar alternatives, (but their effect may be blocked when they would lead to an 'exactly' meaning,) gets all the patterns.

Ignorance implicatures A good number of the existing theories also check off the ignorance implicatures column. However, they again do so to varying degrees, and none completely. The reason is again usually the same: All try to capture ignorance for SMs (mostly focusing on their manifestation in unembedded environments), and none take into account the fact that CMs can give rise to ignorance implicatures also. Geurts and Nouwen (2007) and Nouwen (2010) try to capture the epistemic effect of SMs (not exactly the same as ignorance in the sense of the pragmatic theories) by hardwiring a modal in the meaning of SMs.[23] This leads to complications with embedding and also fails to capture the fact that when embedded under a universal modal the ignorance inferences of SMs become optional. Büring (2008), Coppock and Brochhagen (2013), Kennedy (2015), and Spector (2015) derive the ignorance inferences of SMs as implicatures, and only the latter properly addresses their obligatoriness. Neither the modal nor the existing pragmatic theories capture the fact that CMs can give rise to ignorance inferences also. Moreover, except for Spector's account all the theories are set up such that CMs *cannot* give rise to these inferences. Spector's account could be extended to derive ignorance implicatures for CMs also but, as discussed in §3.12, it begs the question as to why SMs and CMs should have domain alternatives of the type they do, but not BNs. Like the pragmatic approaches, our account also treats ignorance inferences as implicatures, but derives them in a principled way for CMs and SMs but not BNs, and gets the difference in status between the ignorance implicatures of CMs and SMs from the status of their subdomain alternatives, itself derived from the meaning of the comparative as opposed to that of the superlative morpheme.

Acceptability in downward-entailing environments Only a handful of theories check off the acceptability in downward-entailing environments column, and none completely. Geurts and Nouwen (2007) argue that SMs contain an epistemic modal in their meaning, and that epistemic modals resist embedding, which captures infelicity under negation but gives the wrong result for the antecedent of conditionals or the restriction of universals. Cohen and Krifka (2014) argue that only the evaluative meaning of SMs can appear in downward-entailing environments, which gets us the sketch of a solution for antecedents and restrictors, but fails to explain why the evaluative meaning is still bad under negation. And Spector (2015) argues that SMs come with symmetric domain alternatives and a requirement to be exhaustified, which derives infelicity under negation, but also incorrectly predicts infelicity in antecedents/restrictors. None of these accounts thus capture all the patterns. Our account refines and expands on Spector's strategy, deriving instead of merely stipulating the form and status of these alternatives of SMs, and also capturing the acceptability of SMs in the antecedent of a conditional or the restriction of a universal by working out the effect of presupposition on exhaustification following suggestions from Chierchia (2013), Spector (2014), and Nicolae (2017).

Compositionality, predicative uses, constituency Most theories do not explicitly address these issues. In contrast, our account crucially uses the morphological pieces in BNs, CMs, and SMs. We additionally offer a way to handle the predicative uses, and also a plausible constituency.

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[23]On Nouwen (2010)'s account the insertion of the modal is in fact a last resort mechanism. However, since it is triggered whenever the meaning of the SM without it would be equivalent to that of a BN and since this happens virtually every time, this last resort modal is effectively part of the truth conditions for SMs.

To sum up, while the present theory uses many of the insights from the existing theories, it nonetheless improves both empirically and conceptually on them all.

4   Conclusion and open issues

In this paper I outlined a number of basic desiderata for a theory of bare and modified numerals. More specifically, I argued that any theory needs to capture their patterns with respect to entailments, scalar implicatures, ignorance implicatures, and acceptability in downward-entailing environments. I proposed a unified theory of bare and modified numerals that meets all these desiderata, and also other desiderata such as accounting for their compositionality, their predicative uses, and their constituency. I showed that this theory not only has better empirical coverage than any of the existing theories but is also more economical, deriving all the facts from the morphological pieces of these items, on the one hand, and their interaction with general and independently motivated pragmatic mechanisms, on the other.

On this proposal the morphological pieces of bare and modified numerals link up naturally to their manifestations elsewhere. This is an advantage of this theory but also one of the areas where a lot more research is needed. For example, this is immediately obvious for the comparative and superlative modifiers, which have other uses that we acknowledged but did not account for (e.g., cases where the modifier combines with things other than numerals, e.g., *more than happy* or *at least John and Mary*), or for the comparative and the superlative morpheme, whose manifestations elsewhere (e.g., in adjectives) could justifiably be slightly different.

Another attractive feature of this theory is a clearer integration of bare and modified numerals in a system of alternative-activating items. BNs, CMs, and SMs are all items that activate scalar alternatives; CMs and SMs are also items that activate subdomain alternatives that, when used, must lead to proper strengthening; and SMs are furthermore items whose subdomain alternatives are obligatory. This way of looking at BNs, CMs, and SMs in terms of alternatives and exhaustification helps us draw comparisons between them and other items with similar specifications. For example, Spector (2015) argues that these specifications for SMs not only give us the obligativity of their ignorance implicatures in unembedded contexts and their infelicity under negation but also help us make sense of the same signature properties in other items from the broader class of the so-called positive polarity items. (For some concrete examples see the analysis of *qualunque* by Chierchia 2013, of the French disjunction *soit . . . soit* by Spector 2014, or of the French disjunction *ou* by Nicolae 2017.) This connection to the polarity system is both exciting and in need of further research.

Finally, the present theory relies on certain stipulations that could benefit from further investigation. For example, while the observation that the scalar implicatures of BNs, CMs, and SMs are absent only when they would yield an *exactly* meaning for anything other than a plain BN seems correct, it remains to be seen whether the solution proposed here (a blocking effect for BNs, a clash with implicatures from subdomain alternatives, e.g., ignorance, for CMs and SMs) captures all the cases. Also, while making the subdomain alternatives of SMs obligatory is crucial to deriving all the right patterns the way we did, whether their obligatoriness should be derived from the domain-size presupposition of the superlative morpheme is up for debate. That this obligatorily active status of the alternatives should be somehow derived from the superlative morpheme seems desirable: we wouldn't want to say that it is a random lexical choice, or else we should be able to find CMs exhibiting this behavior (obligatory ignorance, infelicity under negation), but, insofar as I am aware, we don't. However, whether tying the obligatoriness of an implicature to a presupposition is more generally attested remains to be seen.

To sum up, the present theory succeeds better than existing theories at explaining the patterns of similarity and difference among bare, comparative-modified, and superlative-modified numerals. It moreover excitingly gives them meanings that link them up naturally to larger phenomena such as comparative and superlative modification or the polarity system. This, however, opens up both new directions and new questions.

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